

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 1
Role: 2026 structure reinforcement

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

This paper is part of the REVISED_HISTORICAL_COVERAGE series. It is calibrated to the 2026 overall difficulty but uses ability combinations from the 2020–2026 official papers. Start each question on a new sheet. Give complete reasons and state theorem hypotheses.

Question 1

For $n \geq 1$ define

$$a_n = \log(n+1) - \log n = \log\left(1 + \frac{1}{n}\right).$$

- (a) (i) Show that $a_n > 0$, that (a_n) is decreasing, and that $a_n \rightarrow 0$. **[3 marks]**
(ii) Using the Mean Value Theorem and the ε - N definition, prove that $na_n \rightarrow 1$. **[3 marks]**
- (b) Determine whether $\sum_{n=1}^{\infty} a_n$ converges. Justify your answer. **[3 marks]**
- (c) Prove that $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges. If its sum is s and its N th partial sum is s_N , prove that $|s - s_N| \leq a_{N+1}$. **[5 marks]**
- (d) Let $F(z) = \sum_{n=1}^{\infty} a_n z^n$.
(i) Prove that the radius of convergence is 1. **[3 marks]**
(ii) Determine convergence at every point of $|z| = 1$, distinguishing $z = 1$ from $z \neq 1$. **[3 marks]**

Question 2

- (a) Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous and suppose that $f(x) = 0$ for every irrational $x \in [0, 1]$. Prove that $f \equiv 0$. **[5 marks]**
- (b) Let $f : (0, 1) \rightarrow \mathbb{R}$ be continuous and suppose that the finite one-sided limits $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$ exist.
- (i) Construct a continuous extension $F : [0, 1] \rightarrow \mathbb{R}$ and prove continuity at the end-points. **[4 marks]**
- (ii) Deduce that f is uniformly continuous on $(0, 1)$. **[4 marks]**
- (c) Let $g : (0, 1) \rightarrow \mathbb{R}$ be $g(x) = \sin(1/x)$.
- (i) Show that g is continuous and bounded. **[2 marks]**
- (ii) Prove that g is not uniformly continuous. **[5 marks]**

Question 3

- (a) For each statement, say whether it is true or false and give a short proof or counterexample.
- (i) If $f, g : [0, 1] \rightarrow \mathbb{R}$ are Darboux integrable, then $\min\{f, g\}$ is Darboux integrable. **[3 marks]**
 - (ii) If f and g are concave on an interval, then $\max\{f, g\}$ is concave. **[3 marks]**
 - (iii) Let $f : I \rightarrow J$ be a bijection between open intervals, with inverse $h : J \rightarrow I$. If both f and h are differentiable, then $f'(x) \neq 0$ for every $x \in I$. **[3 marks]**
 - (iv) Define $u(0) = 0$ and $u(x) = x^4 \sin(1/x^2)$ for $x \neq 0$. Then u is twice differentiable at 0. **[3 marks]**
- (b) State precisely:
- (i) the definition of uniform continuity on a set $X \subset \mathbb{R}$; **[2 marks]**
 - (ii) Taylor's theorem of order n with integral remainder; **[3 marks]**
 - (iii) the Darboux ε -criterion for integrability. **[3 marks]**

Question 4

(a) Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded and continuous except at finitely many points. Prove from the Darboux criterion that f is integrable. **[4 marks]**

(b) Let $f : [0, \infty) \rightarrow \mathbb{R}$ be continuous, twice differentiable on $(0, \infty)$, bounded, and satisfy $f''(x) \leq 0$ for all $x > 0$.

(i) Prove that f is non-decreasing. **[3 marks]**

(ii) Prove from the definition of supremum that $\lim_{x \rightarrow \infty} f(x)$ exists and is finite. **[3 marks]**

(iii) Prove that $\lim_{x \rightarrow \infty} f'(x) = 0$. **[4 marks]**

(iv) Prove that for $x > 0$,

$$xf'(2x) \leq f(2x) - f(x) \leq xf'(x).$$

[4 marks]

(v) Deduce that $\lim_{x \rightarrow \infty} xf'(x) = 0$.

[2 marks]

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 2
Role: 2025 historical bridge

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

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Question 1

(a) Let

$$S = \{(a_n)_{n \geq 1} : a_n \in \mathbb{Q} \text{ for all } n, a_n \rightarrow 1\}.$$

Prove that S is uncountable.

[6 marks]

(b) Let (a_n) be non-negative and $\sum_{n=1}^{\infty} a_n < \infty$. If $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is injective, prove that $\sum_{k=1}^{\infty} a_{\phi(k)}$ converges.

[4 marks]

(c) Suppose that $\{n : a_n \in [-2, 3]\}$ is infinite. Prove that (a_n) has a convergent subsequence.

[3

marks]

(d) Let $s_n = \sum_{j=1}^n a_j$, where $a_n \rightarrow 0$. Suppose that for fixed $\alpha < \beta$ there are infinitely many n with $s_n \leq \alpha$ and infinitely many n with $s_n \geq \beta$. Prove that for every $r \in [\alpha, \beta]$ there is a subsequence $s_{n_k} \rightarrow r$.

[7 marks]

Question 2

- (a) (i) State what it means for a real sequence to be bounded. **[3 marks]**
(ii) State what it means for a real sequence to be Cauchy. **[3 marks]**
(iii) Using only these definitions, prove that every Cauchy sequence is bounded. **[4 marks]**
- (b) Prove from the ε - N definition that if $a_n \rightarrow a$ and $b_n \rightarrow b$, then $a_n b_n \rightarrow ab$. **[5 marks]**
- (c) Suppose (a_n) is bounded and every convergent subsequence of (a_n) has limit a . Prove that $a_n \rightarrow a$. **[5 marks]**

Question 3

- (a) Suppose $\sum a_n$ converges but $\sum a_n^2$ diverges. Prove that $\sum a_n$ is conditionally convergent. **[5 marks]**

- (b) If $a_n \rightarrow a$ and $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is a bijection, prove that $a_{\phi(n)} \rightarrow a$. **[5 marks]**

- (c) Define

$$b_n = \sum_{j=n+1}^{2n} \frac{1}{j+2}.$$

Prove that (b_n) converges.

[5 marks]

- (d) If $\sum_{n=1}^{\infty} a_n$ converges, prove that

$$\frac{1}{n} \sum_{j=1}^n j a_j \rightarrow 0.$$

[5 marks]

Question 4

- (a) Say whether each statement is true or false, giving one line of justification.
- (i) The union of two open subsets of $\mathbb{R}$ is open. **[1 marks]**
 - (ii) If a continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ has infinitely many zeros in every neighbourhood of 0, then f is identically zero on some neighbourhood of 0. **[1 marks]**
 - (iii) Every convex $f : [-1, 1] \rightarrow \mathbb{R}$ is continuous at both endpoints. **[1 marks]**
 - (iv) If f is differentiable on $(-1, 1)$ and $f(-1) < f(1)$, then $f'(x) > 0$ for some $x \in (-1, 1)$. **[1 marks]**
- (b) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing and $S(x_0) = \sup\{f(y) : y < x_0\}$, prove from the definition of supremum that $\lim_{x \rightarrow x_0^-} f(x) = S(x_0)$. **[5 marks]**
- (c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable and suppose $\lim_{x \rightarrow \pm\infty} f(x) = L$. Prove that f has a critical point. **[4 marks]**
- (d) Let $f : [-1, 1] \rightarrow \mathbb{R}$ be Darboux integrable and define $g = f$ except $g(0) = f(0) + 1$. Prove that g is integrable. **[4 marks]**
- (e) Let $f : [-1, 1] \rightarrow \mathbb{R}$ be continuous and non-negative. Prove that $\int_{-1}^1 f = 0$ if and only if $f \equiv 0$. **[3 marks]**

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 3
Role: 2023-2024 historical abilities

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

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Question 1

- (a) Let S be the set of non-increasing functions $f : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$. Prove that S is countably infinite. **[5 marks]**
- (b) Let T be the set of non-decreasing functions $f : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$. Prove that T is uncountable. **[5 marks]**
- (c) Let A, B be non-empty bounded sets of positive real numbers. Prove $\sup(AB) = \sup(A)\sup(B)$, where $AB = \{ab : a \in A, b \in B\}$. **[5 marks]**
- (d) Prove from first principles that if $a_n \rightarrow a \neq 0$, then $1/a_n \rightarrow 1/a$. **[5 marks]**

Question 2

- (a) Let (a_n) be bounded and satisfy $a_{n+1} \geq a_n - 2^{-n-1}$. Prove that (a_n) converges. **[5 marks]**
- (b) Define $x_1 = 1$ and $x_{n+1} = (2 + x_n)/(1 + x_n)$. Prove that (x_n) converges and find its limit. **[7 marks]**
- (c) If $a_n \rightarrow 0$ and $b_n = n^{-1} \sum_{j=1}^n a_j$, prove $b_n \rightarrow 0$. **[6 marks]**
- (d) Give a sequence (a_n) which does not converge but whose arithmetic means b_n do converge. **[2 marks]**

Question 3

- (a) State the ε - δ definition of $\lim_{x \rightarrow a} f(x) = L$. **[2 marks]**
- (b) Prove that $\lim_{x \rightarrow a} f(x) = L$ if and only if $f(x_n) \rightarrow L$ for every sequence $x_n \rightarrow a$ with $x_n \neq a$. **[6 marks]**
- (c) Define $f(x) = x$ for irrational x and $f(x) = -x$ for rational x . Determine exactly for which a the limit $\lim_{x \rightarrow a} f(x)$ exists. **[4 marks]**
- (d) Prove that the Dirichlet function is discontinuous at every real number. **[3 marks]**
- (e) Prove $\lim_{x \rightarrow 0} x \sin(1/x) = 0$. **[2 marks]**
- (f) If $f : [0, 1] \rightarrow [0, 1]$ is continuous, prove that $f(x) = x$ for some x . **[3 marks]**

Question 4

- (a) Let $f : [a, b] \rightarrow \mathbb{R}$ be Darboux integrable and let $t_0 \in (a, b)$. Define $g = f$ except $g(t_0) = f(t_0) + 1$. Prove from Darboux sums that g is integrable and has the same integral as f . **[4 marks]**
- (b) Let $f : [0, 1] \rightarrow \mathbb{R}$ be non-decreasing. Using the uniform partition $P_n = \{0, 1/n, \dots, 1\}$, prove that f is Darboux integrable. **[8 marks]**
- (c) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and has finite limits at $\pm\infty$, prove that f is bounded. **[4 marks]**
- (d) If in addition the two limits are equal and f is differentiable, prove that f has a critical point. **[4 marks]**

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 4
Role: 2020-2022 historical abilities

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

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Question 1

- (a) Let S be all sequences with terms in $\{0, 1, 2\}$; let $T \subset S$ consist of sequences eventually equal to 2; and for fixed N , let $U_N \subset S$ consist of sequences equal to 2 for every $n \geq N$. Classify S, T, U_N as finite, countably infinite, or uncountable, with proof. **[5 marks]**
- (b) For a real sequence (a_n) identify the property expressed by each statement:
- (i) $\exists a \forall \varepsilon > 0 \exists N \forall n \geq N : |a_n - a| < \varepsilon$;
 - (ii) $\forall N \exists a \forall n \geq N : a_n = a$;
 - (iii) $\exists M > 0 \forall n : |a_n| \leq M$;
 - (iv) $\exists a \forall N \forall \varepsilon > 0 \exists n \geq N : |a_n - a| < \varepsilon$.
- [4 marks]**
- (c) Let $A = \{(-1)^n(1 - 1/n) : n \geq 1\}$. Prove that $\sup A = 1$. **[4 marks]**
- (d) If (a_n) is unbounded, prove that it has a subsequence (a_{n_k}) such that $1/a_{n_k} \rightarrow 0$. **[7 marks]**

Question 2

- (a) (i) Prove $\sqrt{n+1} - \sqrt{n} \leq 1/(2\sqrt{n})$ for $n \geq 1$. **[2 marks]**
(ii) Using this inequality, prove from first principles that $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges. **[6 marks]**

- (b) Let $a_n \geq 0$ and $a_n \rightarrow a \in [0, 1)$. Prove that $\sum_{n=1}^{\infty} a_n^n$ converges. **[6 marks]**

- (c) Define

$$c_n = \begin{cases} n^{-2} + n^{-1}, & n \text{ even,} \\ -n^{-2}, & n \text{ odd.} \end{cases}$$

Determine whether $\sum c_n$ converges, and find with proof the radius of convergence of $\sum c_n z^n$. **[6 marks]**

Question 3

- (a) For integer $k \geq 1$ define $f_k(0) = 0$ and $f_k(x) = x^k \sin(1/x^3)$ for $x \neq 0$. Prove continuity at 0 for every k , and differentiability at 0 for every $k \geq 2$, finding $f'_k(0)$. **[5 marks]**
- (b) (i) Prove that $4x = \cos x + 2$ has a real solution. **[2 marks]**
(ii) A continuous $f : [0, 10] \rightarrow \mathbb{R}$ satisfies $f(m) = (-1)^m$ for every integer $0 \leq m \leq 10$. Find the least number of zeros that must occur and justify it. **[3 marks]**
- (c) Let $h(x) = \sqrt{\cos x}$ on $[-\pi/2, \pi/2]$.
(i) Find the second-order Taylor polynomial at 0. **[3 marks]**
(ii) Prove that $\frac{1}{2}(h(0) + h(x)) \leq h(x/2)$ for $0 < x < \pi/2$. **[3 marks]**
- (d) If $f \in C^4([a, b])$ has five distinct zeros, prove that $f^{(4)}(c) = 0$ for some $c \in (a, b)$. **[4 marks]**

Question 4

- (a) Let $f(x) = \log x$ and $P_n = (1, e^{1/n}, \dots, e)$. Write $L(f, P_n)$ and $U(f, P_n)$ explicitly, compute $U - L$, and prove $\lim_{n \rightarrow \infty} U(f, P_n) = 1$. **[8 marks]**
- (b) Determine convergence of $\int_1^\infty (e^{1/x} - 1)^r dx$ when $r = 1$, and prove convergence for every real $r > 1$. **[6 marks]**
- (c) Suppose $f(x) = \sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 1$ and

$$\int_{-1}^1 f(x) x^n dx = 0 \quad (n \geq 0).$$

Prove that $f \equiv 0$ on $[-1, 1]$.

[6 marks]

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 5
Role: seven-year gap-closing

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

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Question 1

- (a) State precisely what it means for $\sum a_n$ to diverge to $+\infty$. **[3 marks]**
- (b) A sequence has infinitely many positive and negative terms. The series of all positive terms diverges to $+\infty$, while the series of all negative terms converges. Prove that every rearrangement of the original series diverges to $+\infty$. **[7 marks]**
- (c) Let $a_n > 0$, $\sum a_n < \infty$, and $r_n = \sum_{j=n}^{\infty} a_j$. Prove that $\sum a_n/\sqrt{r_n}$ converges. **[4 marks]**
- (d) With the same notation, prove that $\sum a_n/r_n$ diverges. **[6 marks]**

Question 2

- (a) Classify each set as open, closed, both, or neither: $(0, 1)$, $\mathbb{Z}$, $\mathbb{Q}$, and $\{0\} \cup [2, \infty)$. Justify briefly. **[5 marks]**
- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and strictly increasing. Prove that $f(U)$ is open whenever $U \subset \mathbb{R}$ is open. **[5 marks]**
- (c) Let $f : [1, \infty) \rightarrow \mathbb{R}$ be uniformly continuous and $f(1) = 0$. Prove that $f(x)/x$ is bounded. **[5 marks]**
- (d) Prove that $g(x) = \sin(x^2)$ is not uniformly continuous on $\mathbb{R}$. **[5 marks]**

Question 3

- (a) Give examples of differentiable functions with each property: (i) $f'(0) = 0$ but no local extremum at 0; (ii) $f'(x) \rightarrow 0$ but $f(x) \rightarrow \infty$; (iii) f bounded but f' unbounded. **[6 marks]**
- (b) If f has $n + 1$ distinct points with the same function value, prove that $f^{(n)}$ vanishes somewhere, assuming all required derivatives exist. **[4 marks]**
- (c) Use the Mean Value Theorem to prove: (i) $f' \equiv c \Rightarrow f(x) = cx + f(0)$; (ii) if $f(x) = 3x$ has two distinct solutions, then $f'(x) = 3$ has a solution; (iii) $x/(1+x^2) < \arctan x < x$ for $x > 0$. **[6 marks]**
- (d) For $F(x) = 1/(1-x^3)$ on $(-1, 1)$, compute $F^{(12)}(0)/12!$ and prove $[F(0) + F(x)]/2 \geq F(x/2)$ for $0 < x < 1$. **[4 marks]**

Question 4

- (a) Let f be twice differentiable and satisfy $f'' + gf' - f = 0$ on $[a, b]$, where g is any function. If $f(a) = f(b) = 0$, prove $f \equiv 0$. **[5 marks]**

- (b) Let f be continuous on $[a, b]$ and suppose $\int_a^b f(x)h(x) dx = 0$ for every continuous h with $h(a) = h(b) = 0$. Prove $f \equiv 0$. **[6 marks]**

- (c) For continuous f , prove

$$\int_0^x (x-t)f(t) dt = \int_0^x \left(\int_0^u f(t) dt \right) du.$$

[5 marks]

- (d) Let $p(0) = 1$ and $p(x) = 0$ for $x \neq 0$ on $[-1, 1]$. Prove p is Darboux integrable and compute its integral. **[4 marks]**

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 6
Role: seven-year mixed transfer pressure

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

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Question 1

- (a) Prove that the set of functions $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ which are periodic in the first variable is uncountable. **[4 marks]**

- (b) Let $A \subset (0, \infty)$ be non-empty and bounded above, and $B \subset (1, \infty)$ non-empty. Prove

$$\sup\{a/b : a \in A, b \in B\} = \frac{\sup A}{\inf B}.$$

[5 marks]

- (c) If $a_n \rightarrow a$ and $b_n \rightarrow b$, prove from first principles that $\max(a_n, b_n) \rightarrow \max(a, b)$. **[5 marks]**

- (d) Let $a_n \geq 0$ and $a_{n+1} \leq a_n + 1/n^2$. Prove that (a_n) converges. **[6 marks]**

Question 2

- (a) Power series $\sum a_n z^n$ and $\sum b_n z^n$ have radii $R_1, R_2 \in (0, \infty)$. Prove that the radius of $\sum a_n b_n z^n$ is at least $R_1 R_2$. **[6 marks]**
- (b) If $\sum a_n$ converges, prove that $\sum a_n/n^2$ converges. **[3 marks]**
- (c) Let $a_n > 0$ and $\sum a_n < \infty$. Put $b_n = n^{-1} \sum_{j=1}^n a_j$. Prove that $\sum b_n$ diverges. **[4 marks]**
- (d) Suppose $a_n \rightarrow a > 0$ and fix $k \geq 1$.
- (i) Prove $(a_{n+1} \cdots a_{n+k})^{1/k} \rightarrow a$. **[3 marks]**
 - (ii) Give a divergent positive sequence for which these geometric means converge when $k = 2$. **[2 marks]**
 - (iii) State the alternating series test. **[2 marks]**

Question 3

- (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be non-decreasing. Prove that its left limit at every point exists and equals $\sup\{f(y) : y < x_0\}$. **[5 marks]**
- (b) If $f : [0, 1] \rightarrow [0, 1]$ is continuous, prove it has a fixed point. **[3 marks]**
- (c) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and even, prove $f'(0) = 0$. **[3 marks]**
- (d) If $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow \infty} f'(x)$ both exist and the first is finite, prove the second limit is 0. **[4 marks]**
- (e) Prove that x^2 is not uniformly continuous on $(0, \infty)$. **[5 marks]**

Question 4

- (a) Prove that a bounded function on $[a, b]$ with finitely many discontinuities is Darboux integrable. **[5 marks]**
- (b) If $f : [-1, 1] \rightarrow \mathbb{R}$ is continuous, non-negative and $\int_{-1}^1 f = 0$, prove $f \equiv 0$. **[3 marks]**
- (c) Let $f \in C^\infty(-1, 1)$ and suppose $|f^{(n)}(x)| \leq (n-1)!$ for all $n \geq 1$. Prove that its Taylor polynomials at 0 converge uniformly to f . **[4 marks]**
- (d) A C^3 function meets the chord joining $(a, f(a))$ and $(b, f(b))$ at two distinct interior points. Prove $f^{(3)}(c) = 0$ for some $c \in (a, b)$. **[4 marks]**
- (e) Let $f : [0, \infty) \rightarrow \mathbb{R}$ be continuous and $f(x)/x \rightarrow 1$. Prove

$$\frac{1}{x^2} \int_0^x f(t) dt \rightarrow \frac{1}{2}.$$

[4 marks]