

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 1
Role: 2026 structure reinforcement

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

This paper is part of the REVISED_HISTORICAL_COVERAGE series. It is calibrated to the 2026 overall difficulty but uses ability combinations from the 2020-2026 official papers. Start each question on a new sheet. Give complete reasons and state theorem hypotheses.

Question 1

For $n \geq 1$ define

$$a_n = \log(n+1) - \log n = \log\left(1 + \frac{1}{n}\right).$$

- (a) (i) Show that $a_n > 0$, that (a_n) is decreasing, and that $a_n \rightarrow 0$. **[3 marks]**
(ii) Using the Mean Value Theorem and the ε - N definition, prove that $na_n \rightarrow 1$. **[3 marks]**
- (b) Determine whether $\sum_{n=1}^{\infty} a_n$ converges. Justify your answer. **[3 marks]**
- (c) Prove that $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges. If its sum is s and its N th partial sum is s_N , prove that $|s - s_N| \leq a_{N+1}$. **[5 marks]**
- (d) Let $F(z) = \sum_{n=1}^{\infty} a_n z^n$.
(i) Prove that the radius of convergence is 1. **[3 marks]**
(ii) Determine convergence at every point of $|z| = 1$, distinguishing $z = 1$ from $z \neq 1$. **[3 marks]**

Question 2

- (a) Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous and suppose that $f(x) = 0$ for every irrational $x \in [0, 1]$. Prove that $f \equiv 0$. **[5 marks]**
- (b) Let $f : (0, 1) \rightarrow \mathbb{R}$ be continuous and suppose that the finite one-sided limits $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$ exist.
- (i) Construct a continuous extension $F : [0, 1] \rightarrow \mathbb{R}$ and prove continuity at the end-points. **[4 marks]**
- (ii) Deduce that f is uniformly continuous on $(0, 1)$. **[4 marks]**
- (c) Let $g : (0, 1) \rightarrow \mathbb{R}$ be $g(x) = \sin(1/x)$.
- (i) Show that g is continuous and bounded. **[2 marks]**
- (ii) Prove that g is not uniformly continuous. **[5 marks]**

Question 3

- (a) For each statement, say whether it is true or false and give a short proof or counterexample.
- (i) If $f, g : [0, 1] \rightarrow \mathbb{R}$ are Darboux integrable, then $\min\{f, g\}$ is Darboux integrable. **[3 marks]**
 - (ii) If f and g are concave on an interval, then $\max\{f, g\}$ is concave. **[3 marks]**
 - (iii) Let $f : I \rightarrow J$ be a bijection between open intervals, with inverse $h : J \rightarrow I$. If both f and h are differentiable, then $f'(x) \neq 0$ for every $x \in I$. **[3 marks]**
 - (iv) Define $u(0) = 0$ and $u(x) = x^4 \sin(1/x^2)$ for $x \neq 0$. Then u is twice differentiable at 0. **[3 marks]**
- (b) State precisely:
- (i) the definition of uniform continuity on a set $X \subset \mathbb{R}$; **[2 marks]**
 - (ii) Taylor's theorem of order n with integral remainder; **[3 marks]**
 - (iii) the Darboux ε -criterion for integrability. **[3 marks]**

Question 4

- (a) Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded and continuous except at finitely many points. Prove from the Darboux criterion that f is integrable. **[4 marks]**
- (b) Let $f : [0, \infty) \rightarrow \mathbb{R}$ be continuous, twice differentiable on $(0, \infty)$, bounded, and satisfy $f''(x) \leq 0$ for all $x > 0$.
- (i) Prove that f is non-decreasing. **[3 marks]**
- (ii) Prove from the definition of supremum that $\lim_{x \rightarrow \infty} f(x)$ exists and is finite. **[3 marks]**
- (iii) Prove that $\lim_{x \rightarrow \infty} f'(x) = 0$. **[4 marks]**
- (iv) Prove that for $x > 0$,
- $$xf'(2x) \leq f(2x) - f(x) \leq xf'(x).$$
- [4 marks]**
- (v) Deduce that $\lim_{x \rightarrow \infty} xf'(x) = 0$. **[2 marks]**