

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 2
Role: 2025 historical bridge

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

This paper is part of the REVISED_HISTORICAL_COVERAGE series. It is calibrated to the 2026 overall difficulty but uses ability combinations from the 2020–2026 official papers. Start each question on a new sheet. Give complete reasons and state theorem hypotheses.

Question 1

(a) Let

$$S = \{(a_n)_{n \geq 1} : a_n \in \mathbb{Q} \text{ for all } n, a_n \rightarrow 1\}.$$

Prove that S is uncountable.

[6 marks]

(b) Let (a_n) be non-negative and $\sum_{n=1}^{\infty} a_n < \infty$. If $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is injective, prove that $\sum_{k=1}^{\infty} a_{\phi(k)}$ converges.

[4 marks]

(c) Suppose that $\{n : a_n \in [-2, 3]\}$ is infinite. Prove that (a_n) has a convergent subsequence.

[3

marks]

(d) Let $s_n = \sum_{j=1}^n a_j$, where $a_n \rightarrow 0$. Suppose that for fixed $\alpha < \beta$ there are infinitely many n with $s_n \leq \alpha$ and infinitely many n with $s_n \geq \beta$. Prove that for every $r \in [\alpha, \beta]$ there is a subsequence $s_{n_k} \rightarrow r$.

[7 marks]

Question 2

- (a) (i) State what it means for a real sequence to be bounded. **[3 marks]**
(ii) State what it means for a real sequence to be Cauchy. **[3 marks]**
(iii) Using only these definitions, prove that every Cauchy sequence is bounded. **[4 marks]**
- (b) Prove from the ε - N definition that if $a_n \rightarrow a$ and $b_n \rightarrow b$, then $a_n b_n \rightarrow ab$. **[5 marks]**
- (c) Suppose (a_n) is bounded and every convergent subsequence of (a_n) has limit a . Prove that $a_n \rightarrow a$. **[5 marks]**

Question 3

- (a) Suppose $\sum a_n$ converges but $\sum a_n^2$ diverges. Prove that $\sum a_n$ is conditionally convergent. **[5 marks]**

- (b) If $a_n \rightarrow a$ and $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is a bijection, prove that $a_{\phi(n)} \rightarrow a$. **[5 marks]**

- (c) Define

$$b_n = \sum_{j=n+1}^{2n} \frac{1}{j+2}.$$

Prove that (b_n) converges.

[5 marks]

- (d) If $\sum_{n=1}^{\infty} a_n$ converges, prove that

$$\frac{1}{n} \sum_{j=1}^n j a_j \rightarrow 0.$$

[5 marks]

Question 4

- (a) Say whether each statement is true or false, giving one line of justification.
- (i) The union of two open subsets of $\mathbb{R}$ is open. **[1 marks]**
 - (ii) If a continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ has infinitely many zeros in every neighbourhood of 0, then f is identically zero on some neighbourhood of 0. **[1 marks]**
 - (iii) Every convex $f : [-1, 1] \rightarrow \mathbb{R}$ is continuous at both endpoints. **[1 marks]**
 - (iv) If f is differentiable on $(-1, 1)$ and $f(-1) < f(1)$, then $f'(x) > 0$ for some $x \in (-1, 1)$. **[1 marks]**
- (b) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing and $S(x_0) = \sup\{f(y) : y < x_0\}$, prove from the definition of supremum that $\lim_{x \rightarrow x_0^-} f(x) = S(x_0)$. **[5 marks]**
- (c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable and suppose $\lim_{x \rightarrow \pm\infty} f(x) = L$. Prove that f has a critical point. **[4 marks]**
- (d) Let $f : [-1, 1] \rightarrow \mathbb{R}$ be Darboux integrable and define $g = f$ except $g(0) = f(0) + 1$. Prove that g is integrable. **[4 marks]**
- (e) Let $f : [-1, 1] \rightarrow \mathbb{R}$ be continuous and non-negative. Prove that $\int_{-1}^1 f = 0$ if and only if $f \equiv 0$. **[3 marks]**