

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 3
Role: 2023-2024 historical abilities

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

This paper is part of the REVISED_HISTORICAL_COVERAGE series. It is calibrated to the 2026 overall difficulty but uses ability combinations from the 2020–2026 official papers. Start each question on a new sheet. Give complete reasons and state theorem hypotheses.

Question 1

- (a) Let S be the set of non-increasing functions $f : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$. Prove that S is countably infinite. **[5 marks]**
- (b) Let T be the set of non-decreasing functions $f : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$. Prove that T is uncountable. **[5 marks]**
- (c) Let A, B be non-empty bounded sets of positive real numbers. Prove $\sup(AB) = \sup(A)\sup(B)$, where $AB = \{ab : a \in A, b \in B\}$. **[5 marks]**
- (d) Prove from first principles that if $a_n \rightarrow a \neq 0$, then $1/a_n \rightarrow 1/a$. **[5 marks]**

Question 2

- (a) Let (a_n) be bounded and satisfy $a_{n+1} \geq a_n - 2^{-n-1}$. Prove that (a_n) converges. **[5 marks]**
- (b) Define $x_1 = 1$ and $x_{n+1} = (2 + x_n)/(1 + x_n)$. Prove that (x_n) converges and find its limit. **[7 marks]**
- (c) If $a_n \rightarrow 0$ and $b_n = n^{-1} \sum_{j=1}^n a_j$, prove $b_n \rightarrow 0$. **[6 marks]**
- (d) Give a sequence (a_n) which does not converge but whose arithmetic means b_n do converge. **[2 marks]**

Question 3

- (a) State the ε - δ definition of $\lim_{x \rightarrow a} f(x) = L$. **[2 marks]**
- (b) Prove that $\lim_{x \rightarrow a} f(x) = L$ if and only if $f(x_n) \rightarrow L$ for every sequence $x_n \rightarrow a$ with $x_n \neq a$. **[6 marks]**
- (c) Define $f(x) = x$ for irrational x and $f(x) = -x$ for rational x . Determine exactly for which a the limit $\lim_{x \rightarrow a} f(x)$ exists. **[4 marks]**
- (d) Prove that the Dirichlet function is discontinuous at every real number. **[3 marks]**
- (e) Prove $\lim_{x \rightarrow 0} x \sin(1/x) = 0$. **[2 marks]**
- (f) If $f : [0, 1] \rightarrow [0, 1]$ is continuous, prove that $f(x) = x$ for some x . **[3 marks]**

Question 4

- (a) Let $f : [a, b] \rightarrow \mathbb{R}$ be Darboux integrable and let $t_0 \in (a, b)$. Define $g = f$ except $g(t_0) = f(t_0) + 1$. Prove from Darboux sums that g is integrable and has the same integral as f . **[4 marks]**
- (b) Let $f : [0, 1] \rightarrow \mathbb{R}$ be non-decreasing. Using the uniform partition $P_n = \{0, 1/n, \dots, 1\}$, prove that f is Darboux integrable. **[8 marks]**
- (c) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and has finite limits at $\pm\infty$, prove that f is bounded. **[4 marks]**
- (d) If in addition the two limits are equal and f is differentiable, prove that f has a critical point. **[4 marks]**