

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 4
Role: 2020-2022 historical abilities

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

This paper is part of the REVISED_HISTORICAL_COVERAGE series. It is calibrated to the 2026 overall difficulty but uses ability combinations from the 2020–2026 official papers. Start each question on a new sheet. Give complete reasons and state theorem hypotheses.

Question 1

- (a) Let S be all sequences with terms in $\{0, 1, 2\}$; let $T \subset S$ consist of sequences eventually equal to 2; and for fixed N , let $U_N \subset S$ consist of sequences equal to 2 for every $n \geq N$. Classify S, T, U_N as finite, countably infinite, or uncountable, with proof. **[5 marks]**
- (b) For a real sequence (a_n) identify the property expressed by each statement:
- (i) $\exists a \forall \varepsilon > 0 \exists N \forall n \geq N : |a_n - a| < \varepsilon$;
 - (ii) $\forall N \exists a \forall n \geq N : a_n = a$;
 - (iii) $\exists M > 0 \forall n : |a_n| \leq M$;
 - (iv) $\exists a \forall N \forall \varepsilon > 0 \exists n \geq N : |a_n - a| < \varepsilon$.
- [4 marks]**
- (c) Let $A = \{(-1)^n(1 - 1/n) : n \geq 1\}$. Prove that $\sup A = 1$. **[4 marks]**
- (d) If (a_n) is unbounded, prove that it has a subsequence (a_{n_k}) such that $1/a_{n_k} \rightarrow 0$. **[7 marks]**

Question 2

- (a) (i) Prove $\sqrt{n+1} - \sqrt{n} \leq 1/(2\sqrt{n})$ for $n \geq 1$. **[2 marks]**
(ii) Using this inequality, prove from first principles that $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges. **[6 marks]**

- (b) Let $a_n \geq 0$ and $a_n \rightarrow a \in [0, 1)$. Prove that $\sum_{n=1}^{\infty} a_n^n$ converges. **[6 marks]**

- (c) Define

$$c_n = \begin{cases} n^{-2} + n^{-1}, & n \text{ even,} \\ -n^{-2}, & n \text{ odd.} \end{cases}$$

Determine whether $\sum c_n$ converges, and find with proof the radius of convergence of $\sum c_n z^n$. **[6 marks]**

Question 3

- (a) For integer $k \geq 1$ define $f_k(0) = 0$ and $f_k(x) = x^k \sin(1/x^3)$ for $x \neq 0$. Prove continuity at 0 for every k , and differentiability at 0 for every $k \geq 2$, finding $f'_k(0)$. **[5 marks]**
- (b) (i) Prove that $4x = \cos x + 2$ has a real solution. **[2 marks]**
(ii) A continuous $f : [0, 10] \rightarrow \mathbb{R}$ satisfies $f(m) = (-1)^m$ for every integer $0 \leq m \leq 10$. Find the least number of zeros that must occur and justify it. **[3 marks]**
- (c) Let $h(x) = \sqrt{\cos x}$ on $[-\pi/2, \pi/2]$.
(i) Find the second-order Taylor polynomial at 0. **[3 marks]**
(ii) Prove that $\frac{1}{2}(h(0) + h(x)) \leq h(x/2)$ for $0 < x < \pi/2$. **[3 marks]**
- (d) If $f \in C^4([a, b])$ has five distinct zeros, prove that $f^{(4)}(c) = 0$ for some $c \in (a, b)$. **[4 marks]**

Question 4

- (a) Let $f(x) = \log x$ and $P_n = (1, e^{1/n}, \dots, e)$. Write $L(f, P_n)$ and $U(f, P_n)$ explicitly, compute $U - L$, and prove $\lim_{n \rightarrow \infty} U(f, P_n) = 1$. **[8 marks]**
- (b) Determine convergence of $\int_1^\infty (e^{1/x} - 1)^r dx$ when $r = 1$, and prove convergence for every real $r > 1$. **[6 marks]**
- (c) Suppose $f(x) = \sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 1$ and

$$\int_{-1}^1 f(x) x^n dx = 0 \quad (n \geq 0).$$

Prove that $f \equiv 0$ on $[-1, 1]$.

[6 marks]