

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 5
Role: seven-year gap-closing

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

This paper is part of the REVISED_HISTORICAL_COVERAGE series. It is calibrated to the 2026 overall difficulty but uses ability combinations from the 2020–2026 official papers. Start each question on a new sheet. Give complete reasons and state theorem hypotheses.

Question 1

- (a) State precisely what it means for $\sum a_n$ to diverge to $+\infty$. **[3 marks]**
- (b) A sequence has infinitely many positive and negative terms. The series of all positive terms diverges to $+\infty$, while the series of all negative terms converges. Prove that every rearrangement of the original series diverges to $+\infty$. **[7 marks]**
- (c) Let $a_n > 0$, $\sum a_n < \infty$, and $r_n = \sum_{j=n}^{\infty} a_j$. Prove that $\sum a_n/\sqrt{r_n}$ converges. **[4 marks]**
- (d) With the same notation, prove that $\sum a_n/r_n$ diverges. **[6 marks]**

Question 2

- (a) Classify each set as open, closed, both, or neither: $(0, 1)$, $\mathbb{Z}$, $\mathbb{Q}$, and $\{0\} \cup [2, \infty)$. Justify briefly. **[5 marks]**
- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and strictly increasing. Prove that $f(U)$ is open whenever $U \subset \mathbb{R}$ is open. **[5 marks]**
- (c) Let $f : [1, \infty) \rightarrow \mathbb{R}$ be uniformly continuous and $f(1) = 0$. Prove that $f(x)/x$ is bounded. **[5 marks]**
- (d) Prove that $g(x) = \sin(x^2)$ is not uniformly continuous on $\mathbb{R}$. **[5 marks]**

Question 3

- (a) Give examples of differentiable functions with each property: (i) $f'(0) = 0$ but no local extremum at 0; (ii) $f'(x) \rightarrow 0$ but $f(x) \rightarrow \infty$; (iii) f bounded but f' unbounded. **[6 marks]**
- (b) If f has $n + 1$ distinct points with the same function value, prove that $f^{(n)}$ vanishes somewhere, assuming all required derivatives exist. **[4 marks]**
- (c) Use the Mean Value Theorem to prove: (i) $f' \equiv c \Rightarrow f(x) = cx + f(0)$; (ii) if $f(x) = 3x$ has two distinct solutions, then $f'(x) = 3$ has a solution; (iii) $x/(1 + x^2) < \arctan x < x$ for $x > 0$. **[6 marks]**
- (d) For $F(x) = 1/(1 - x^3)$ on $(-1, 1)$, compute $F^{(12)}(0)/12!$ and prove $[F(0) + F(x)]/2 \geq F(x/2)$ for $0 < x < 1$. **[4 marks]**

Question 4

- (a) Let f be twice differentiable and satisfy $f'' + gf' - f = 0$ on $[a, b]$, where g is any function. If $f(a) = f(b) = 0$, prove $f \equiv 0$. **[5 marks]**

- (b) Let f be continuous on $[a, b]$ and suppose $\int_a^b f(x)h(x) dx = 0$ for every continuous h with $h(a) = h(b) = 0$. Prove $f \equiv 0$. **[6 marks]**

- (c) For continuous f , prove

$$\int_0^x (x-t)f(t) dt = \int_0^x \left(\int_0^u f(t) dt \right) du.$$

[5 marks]

- (d) Let $p(0) = 1$ and $p(x) = 0$ for $x \neq 0$ on $[-1, 1]$. Prove p is Darboux integrable and compute its integral. **[4 marks]**