

Imperial College London style training paper
MATH40002 Analysis 1
Revised Mock Examination 6
Role: seven-year mixed transfer pressure

Time allowed: 2 hours

Total: 80 marks

Answer all four questions

This paper is part of the REVISED_HISTORICAL_COVERAGE series. It is calibrated to the 2026 overall difficulty but uses ability combinations from the 2020–2026 official papers. Start each question on a new sheet. Give complete reasons and state theorem hypotheses.

Question 1

- (a) Prove that the set of functions $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ which are periodic in the first variable is uncountable. **[4 marks]**

- (b) Let $A \subset (0, \infty)$ be non-empty and bounded above, and $B \subset (1, \infty)$ non-empty. Prove

$$\sup\{a/b : a \in A, b \in B\} = \frac{\sup A}{\inf B}.$$

[5 marks]

- (c) If $a_n \rightarrow a$ and $b_n \rightarrow b$, prove from first principles that $\max(a_n, b_n) \rightarrow \max(a, b)$. **[5 marks]**

- (d) Let $a_n \geq 0$ and $a_{n+1} \leq a_n + 1/n^2$. Prove that (a_n) converges. **[6 marks]**

Question 2

- (a) Power series $\sum a_n z^n$ and $\sum b_n z^n$ have radii $R_1, R_2 \in (0, \infty)$. Prove that the radius of $\sum a_n b_n z^n$ is at least $R_1 R_2$. **[6 marks]**
- (b) If $\sum a_n$ converges, prove that $\sum a_n/n^2$ converges. **[3 marks]**
- (c) Let $a_n > 0$ and $\sum a_n < \infty$. Put $b_n = n^{-1} \sum_{j=1}^n a_j$. Prove that $\sum b_n$ diverges. **[4 marks]**
- (d) Suppose $a_n \rightarrow a > 0$ and fix $k \geq 1$.
- (i) Prove $(a_{n+1} \cdots a_{n+k})^{1/k} \rightarrow a$. **[3 marks]**
 - (ii) Give a divergent positive sequence for which these geometric means converge when $k = 2$. **[2 marks]**
 - (iii) State the alternating series test. **[2 marks]**

Question 3

- (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be non-decreasing. Prove that its left limit at every point exists and equals $\sup\{f(y) : y < x_0\}$. **[5 marks]**
- (b) If $f : [0, 1] \rightarrow [0, 1]$ is continuous, prove it has a fixed point. **[3 marks]**
- (c) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable and even, prove $f'(0) = 0$. **[3 marks]**
- (d) If $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow \infty} f'(x)$ both exist and the first is finite, prove the second limit is 0. **[4 marks]**
- (e) Prove that x^2 is not uniformly continuous on $(0, \infty)$. **[5 marks]**

Question 4

- (a) Prove that a bounded function on $[a, b]$ with finitely many discontinuities is Darboux integrable. **[5 marks]**
- (b) If $f : [-1, 1] \rightarrow \mathbb{R}$ is continuous, non-negative and $\int_{-1}^1 f = 0$, prove $f \equiv 0$. **[3 marks]**
- (c) Let $f \in C^\infty(-1, 1)$ and suppose $|f^{(n)}(x)| \leq (n-1)!$ for all $n \geq 1$. Prove that its Taylor polynomials at 0 converge uniformly to f . **[4 marks]**
- (d) A C^3 function meets the chord joining $(a, f(a))$ and $(b, f(b))$ at two distinct interior points. Prove $f^{(3)}(c) = 0$ for some $c \in (a, b)$. **[4 marks]**
- (e) Let $f : [0, \infty) \rightarrow \mathbb{R}$ be continuous and $f(x)/x \rightarrow 1$. Prove

$$\frac{1}{x^2} \int_0^x f(t) dt \rightarrow \frac{1}{2}.$$

[4 marks]