

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Analysis 1

Date: Friday, May 16, 2025

Time: Start time 14:00 – End time 17:00 (BST)

Time Allowed: 3 hours

This paper has 6 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Define the set of sequences

$$\mathcal{S} = \left\{ (a_n)_{n=1}^{\infty} : a_n \in \mathbb{Q}, a_n \rightarrow 0 \text{ as } n \rightarrow \infty \right\}.$$

Is the set $\mathcal{S}$ countably infinite or uncountably infinite? Prove your answer. (6 marks)

- (b) Let $(a_n)_{n=1}^{\infty}$ be a non-negative sequence with the property that $\sum_{j=1}^{\infty} a_j$ converges. Carefully prove that, for any subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$, $\sum_{k=1}^{\infty} a_{n_k}$ converges. (4 marks)

- (c) Suppose that $(a_n)_{n=1}^{\infty}$ is a subsequence such that $\{n \in \mathbb{N}_{>0} : a_n \in [-1, 1]\}$ is infinite. Carefully argue that $(a_n)_{n=1}^{\infty}$ has a convergent subsequence. (3 marks)

- (d) Let $(a_n)_{n=1}^{\infty}$ be a sequence with the following properties.

* $\lim_{n \rightarrow \infty} a_n = 0,$

* There are infinitely many $n \in \mathbb{N}_{>0}$ such that $\sum_{j=1}^n a_j \leq 0.$

* There are infinitely many $n \in \mathbb{N}_{>0}$ such that $\sum_{j=1}^n a_j \geq 1.$

Prove that, for any $r \in [0, 1]$, one can find a sequence $n_k \rightarrow \infty$ as $k \rightarrow \infty$ such that

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{n_k} a_j = r.$$

(7 marks)

(Total: 20 marks)

2. (a) (i) State the definition of “ $(a_n)_{n=1}^{\infty}$ is bounded”. (3 marks)

- (ii) State the definition of “ $(a_n)_{n=1}^{\infty}$ is Cauchy”. (3 marks)

- (iii) Carefully prove, from first principles (that is, only using the two definitions above), that every Cauchy sequence is bounded. (4 marks)

- (b) Prove, from first principles (that is, only using the definition of convergence), that if $a_n \rightarrow a$ and $b_n \rightarrow 0$, then $a_n b_n \rightarrow 0$. (5 marks)

- (c) Suppose that the sequence $(a_n)_{n=1}^{\infty}$ satisfies the following properties:

* $(a_n)_{n=1}^{\infty}$ is bounded.

* There exists $a \in \mathbb{R}$ such that every convergent subsequence of $(a_n)_{n=1}^{\infty}$ must converge to a .

True or False: It must be the case that $a_n \rightarrow a$.

If you answer true, give a proof. If you answer false, give a counterexample with justification. (5 marks)

(Total: 20 marks)

3. (a) For the next two questions, suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers, and suppose that the series $\sum_{n=1}^{\infty} a_n$ converges.

(i) Prove $\sum_{n=1}^{\infty} a_n^2$ divergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ conditionally convergent. (5 marks)

(ii) Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n j \cdot a_j = 0 .$$

(5 marks)

(b) Suppose that $(a_n)_{n=0}^{\infty}$ is a sequence such that $a_n \rightarrow a$ as $n \rightarrow \infty$. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Carefully prove that $a_{f(n)} \rightarrow a$ as $n \rightarrow \infty$. (5 marks)

(c) Define the sequence $(a_n)_{n=1}^{\infty}$ by setting

$$a_n = \sum_{j=n+1}^{2n} \frac{1}{j} .$$

Prove that a_n is convergent.

(5 marks)

(Total: 20 marks)

4. (a) For each statement below say whether it is TRUE or FALSE. When TRUE write a one or two line proof, and when FALSE provide a counter-example without proof. Answering TRUE/FALSE correctly is worth 1 mark, and a proof or counter-example is worth 1 mark.
- (i) Let A and B be non-empty open sets of $\mathbb{R}$ (not necessarily open intervals). Then $A \cup B$ is also an open set. (2 marks)
 - (ii) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous everywhere. Assume that for every $\delta > 0$, f attains the value 0 infinitely many times on $I_\delta = (-\delta, \delta)$, then f must be identically zero on I_δ for some $\delta > 0$. (2 marks)
 - (iii) Let $f : \mathbb{R} \rightarrow \mathbb{R}$. If $\lim_{x \rightarrow a} f(x) = L$, for $L \in \mathbb{R}$, there exist an open interval, containing a , on which f is bounded. (2 marks)
 - (iv) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous everywhere and $g : \mathbb{R} \rightarrow \mathbb{R}$ is discontinuous everywhere, then $g \circ f$ is not continuous anywhere. (2 marks)
- (b) Write an ε and δ definition for “the limit of $f(x)$ when x goes to a , does not exist”. (2 marks)
- (c) Prove from first principles, i.e. using ε and δ and without using the algebra of limits, that

$$\lim_{x \rightarrow 4} \frac{7}{x^2 - 9} = 1.$$

(5 marks)

- (d) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone non-decreasing function. Let $S(x) = \sup\{f(y) : y < x\}$. Prove, from first principles (i.e. using only monotonicity and the definition of supremum) that for all $x_0 \in \mathbb{R}$, the one-sided limit $\lim_{x \rightarrow x_0^-} f(x)$ is equal to $S(x_0)$. (5 marks)

(Total: 20 marks)

5. (a) For each statement below say whether it is TRUE or FALSE. When TRUE write a one or two line proof, and when FALSE provide a counter-example without proof. Answering TRUE/FALSE correctly is worth 1 mark, and a proof or counter-example is worth 1 mark.
- (i) Let $f : [-1, 1] \rightarrow \mathbb{R}$ be continuous on $[-1, 1]$ and twice differentiable on $(-1, 1)$. If $f''(x) > 0$ then f attains a local minimum on $(-1, 1)$. (2 marks)
 - (ii) If $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Intermediate Value Property, then f is continuous everywhere. (2 marks)
 - (iii) If $f : [-1, 1] \rightarrow \mathbb{R}$ is convex, then f is continuous everywhere on $[-1, 1]$. (2 marks)
 - (iv) If $f : [-1, 1] \rightarrow \mathbb{R}$ is differentiable on $(-1, 1)$ and $f(-1) < f(1)$, then there exists $x \in (-1, 1)$ such that $f'(x) > 0$. (2 marks)
- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$. Define, carefully, what it means for $f''(x)$ to exist. (5 marks)
- (c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable everywhere and symmetric, i.e. $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Prove that $f'(0) = 0$. (3 marks)
- (d) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable everywhere and $L \in \mathbb{R}$. Prove that if

$$\lim_{x \rightarrow +\infty} f(x) = L = \lim_{x \rightarrow -\infty} f(x)$$

then f admits a critical point. (4 marks)

(Total: 20 marks)

6. (a) For each statement below say whether it is TRUE or FALSE. When TRUE write a one or two line proof, and when FALSE provide a counter-example without proof. Answering TRUE/FALSE correctly is worth 1 mark, and a proof or counter-example is worth 1 mark.
- (i) Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function with the property that $L(f, P) = U(f, P)$ for each partition P of $[a, b]$. Then f must be constant. (2 marks)
 - (ii) Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function satisfying $f(x) \geq 0$ for all $x \in [a, b]$. Then, $\int_a^b f(x)dx = 0$ implies f is constant. (2 marks)
 - (iii) Assume $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex. Then f is integrable on every closed interval $[a, b]$, for $a, b \in \mathbb{R}$. (2 marks)
 - (iv) Let P be a partition of $[a, b]$ and $x \in [a, b]$. If $P' = P \cup \{x\}$ then for any function $f : [a, b] \rightarrow \mathbb{R}$, it must hold $L(f, P) \leq L(f, P')$. (2 marks)
- (b) Let $f : [-1, 1] \rightarrow \mathbb{R}$ be an integrable function. Let $g(x) = f(x)$ for $x \neq 0$ and $g(0) = f(0) + 1$. Prove that g is integrable. (4 marks)

(c) Prove

$$f(x) = \begin{cases} x + 1 & x < 0 \\ 3 - 2x & x \geq 0 \end{cases}$$

is integrable on $[-1, 1]$ and compute its integral. Justify your steps by citing the theorems proved in the lectures. (5 marks)

- (d) Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a continuous function. Assume $f(x) \geq 0$ for all $x \in [-1, 1]$. Prove that $\int_{-1}^1 f(x)dx = 0$ if and only if $f(x) = 0$. (3 marks)

(Total: 20 marks)

1. (a) The set $\mathcal{S}$ is uncountably infinite. (1 mark) To prove this, we first define the set

$$\mathcal{T} = \{(a_n)_{n=1}^{\infty} : a_n \in \{0, 1/n\}\}.$$

For any $(a_n)_{n=1}^{\infty} \in \tilde{\mathcal{S}}$, we have $|a_n| \leq 1/n \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $\mathcal{T} \subset \mathcal{S}$, and it would suffice to show that $\mathcal{T}$ is uncountably infinite. We prove this last claim using Cantor's diagonal argument (2 marks for reasonable strategy)

We suppose by contradiction that we have an enumeration $\{(a_n^{(m)})_{n=1}^{\infty}\}_{m=1}^{\infty}$ of $\tilde{\mathcal{S}}$. We now claim we can construct a sequence $(b_n)_{n=1}^{\infty}$ not belonging to this enumeration by setting $b_n = 1/n$ if $a_n^{(n)} = 0$ and $b_n = 0$ if $a_n^{(n)} = 1/n$. This sequence $(b_n)_{n=1}^{\infty}$ is not in the enumeration, as it differs from each $(a_n^{(m)})_{n=1}^{\infty}$ in at least one place, that is $b_n \neq a_n^{(n)}$. (3 marks for finishing the proof)

6, C

- (b) Fix a subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$. Since the sequence is non-negative, the sequence of partial sums $s_l = \sum_{k=1}^l a_{n_k}$ monotone increase and to show it converges it suffices to show that $(s_l)_{l=1}^{\infty}$ is bounded above. We claim it is bounded above by $M = \sum_{n=1}^{\infty} a_n < \infty$ since

$$s_l = \sum_{k=1}^l a_{n_k} \leq \sum_{n=1}^{n_l} a_n \leq \sum_{n=1}^{\infty} a_n = M.$$

4, A

- (c) From the assumption, we can find a sequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ such that $a_{n_k} \in [-1, 1]$ for all k , so it is bounded. By Bolzano-Weierstrass, we can find a convergent subsequence $(a_{n_{k_l}})_{l=1}^{\infty}$ of $(a_{n_k})_{k=1}^{\infty}$. However, $(a_{n_{k_l}})_{l=1}^{\infty}$ is also a subsequence of $(a_n)_{n=1}^{\infty}$.

3, A

- (d) We write $s_n = \sum_{j=1}^n a_j$, we want to show that for any $r \in [0, 1]$, there is a subsequence $(s_{n_k})_{k=1}^{\infty}$ with $s_{n_k} \rightarrow r$ as $k \rightarrow \infty$. We define, for each n , $\mathcal{I}_n = \{[0, 1/n], [1/n, 2/n], \dots, [(n-1)/n, 1]\}$. It suffices to find a subsequence with the property that for each k , s_{n_k} and δ belong to a common element of $\mathcal{I}_k$. It then follows that $|s_{n_k} - \delta| \leq \frac{1}{k} \rightarrow 0$ as $k \rightarrow \infty$. (3 marks for identifying a reasonable strategy)

We construct the subsequence by induction, and discuss the inductive step as the base case is essentially the same. Suppose we have chosen $n_1 < \dots < n_k$ as desired, we now need to choose an appropriate $n_{k+1} > n_k$. We note that there exists some $N > M > n_k$ such that,

- * For all $n \geq M$, one has $|s_{n+1} - s_n| = |a_{n+1}| < 1/(k+1)$.
- * $s_M \leq 0$.
- * $s_N \geq 1$.

The justifications for each of the above statements follow from the corresponding assumption in the statement of the problem. It then follows that each element of $[a, b] \in \mathcal{I}_{k+1}$, there exists at least one $N \geq n \geq M$ with $s_n \in [a, b]$. We also know r belongs to at least one element of $\mathcal{I}_{k+1}$, so, as desired, we may choose $N \geq n_{k+1} \geq M$ with $s_{n_{k+1}}$ and r belonging to the same element of $\mathcal{I}_{k+1}$. (4 marks for precise proof)

7, D

2. (a) (i) There exists $M \geq 0$ such that for all $n \in \mathbb{N}_{>0}$, $|a_n| \leq M$.

3, A

(ii) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}_{>0}$ such that for every $n, m \geq N$,

$$|a_n - a_m| < \varepsilon .$$

3, A

(iii) Let $\varepsilon = 1$, then there exists $N \in \mathbb{N}_{>0}$ such that for every $n, m \geq N$,

$$|a_n - a_m| < 1 .$$

We then claim that $(a_n)_{n=1}^{\infty}$ is bounded by $M = \max\{|a_1|, \dots, |a_N| + 1\}$.

Clearly $|a_n| \leq M$ for every $n \leq N$, while for $n > N$ we have

$$|a_n| \leq |a_n - a_N| + |a_N| < 1 + |a_N| \leq M .$$

4, A

(b) Let $\varepsilon > 0$. Since $b_n \rightarrow 0$, there exists $N_1 \in \mathbb{N}_{>0}$ such that for every $n \geq N_1$, $|b_n| < \varepsilon/(|a| + 1)$. Since $a_n \rightarrow a$, there exists $N_2 \in \mathbb{N}_{>0}$ such that for every $n \geq N_2$, $|a_n - a| < 1$, so that $|a_n| \leq |a_n - a| + |a| < |a| + 1$. We then have for every $n \geq N = \max\{N_1, N_2\}$,

$$|a_n b_n| < \varepsilon/(|a| + 1)(|a| + 1) = \varepsilon .$$

5, A

(c) The statement is **True** (1 mark). We prove this by arguing by contradiction. Suppose that $a_n \not\rightarrow a$, this means there exists some $\delta > 0$ such that for every $N \in \mathbb{N}_{>0}$, there exists $n \geq N$ with $|a_n - a| \geq \delta$.

We now construct a subsequence (a_{n_k}) of (a_n) such that for all k , $|a_{n_k} - a| \geq \delta$. By our assumption, there exists some n_1 such that $|a_{n_1} - a| \geq \delta$. Similarly, by taking $N = n_1 + 1$ above, we know there exists $n_2 > n_1$ such that $|a_{n_2} - a| \geq \delta$. Inductively, given $n_1 < n_2 < \dots < n_k$ with $|a_{n_j} - a| \geq \delta$ for $1 \leq j \leq k$, we can find $n_{k+1} > n_k$ (by setting $N = n_k + 1$ above) such that $|a_{n_{k+1}} - a| \geq \delta$. This gives the desired subsequence.

Since (a_n) is bounded, so is (a_{n_k}) and so by the Bolzano-Weierstrass theorem it has its own convergent subsequence $(a_{n_{k_j}})$. (1 mark for invoking Bolzano Weierstrass, or another theorem that gives a reasonable strategy). However, $(a_{n_{k_j}})$ is also a convergent subsequence of (a_n) , so we must have $(a_{n_{k_j}}) \rightarrow a$ as $j \rightarrow \infty$.

Then, by setting $\varepsilon = \delta > 0$, there exists M such that $j \geq M \Rightarrow |a_{n_{k_j}} - a| < \varepsilon = \delta$. On the other hand, we have that $|a_{n_{k_j}} - a| \geq \delta$ for every j since it is a subsequence of (a_{n_k}) - these two inequalities give us our contradiction. (3 marks for finishing the proof)

5, B

5, B

3. (a) (i) We prove the contrapositive, that is if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n^2$ is convergent. Since $\sum_{n=1}^{\infty} a_n$ is convergent, it follows that $\lim_{n \rightarrow \infty} a_n = 0$, in particular the a_n are bounded. Therefore, there exists $M \geq 0$ such that $|a_n| \leq M$ for all n . It follows that $|a_n^2| \leq M|a_n|$ for all n , so that $\sum_{n=1}^{\infty} a_n^2$ is (absolutely) convergent by comparison with $\sum_{n=1}^{\infty} M|a_n|$ which is convergent itself.

5, B

- (ii) We define $s_n = \sum_{j=n}^{\infty} a_j$, it follows that $s_n \rightarrow 0$ as $n \rightarrow \infty$ since the series $\sum_{j=1}^{\infty} a_j$ is convergent. We then observe that

$$\frac{1}{n} \sum_{j=1}^n j a_j = \frac{1}{n} (s_1 + s_2 + \cdots + s_n) = \frac{1}{n} \sum_{j=1}^n s_j.$$

It follows that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n s_j = \lim_{n \rightarrow \infty} s_n = 0$ where the first equality is by a problem sheet question.

5, D

- (b) Let $\varepsilon > 0$, then there exists $M \in \mathbb{N}_{>0}$ such that $n \geq M$ implies that $|a_n - a| < \varepsilon$. Now, choose $N = \max\{i \in \mathbb{N}_{>0} : f(i) < M\} + 1$. It follows that for any $j \geq N$, $f(j) \geq M$. Therefore, for all $j \geq N$, $|a_{f(j)} - a| < \varepsilon$.

5, B

- (c) It suffices to show that $(a_n)_{n=1}^{\infty}$ is increasing and bounded above. To note that it is increasing, we note that

$$\begin{aligned} a_{n+1} - a_n &= \sum_{j=n+2}^{2n+2} \frac{1}{j} - \sum_{j=n+1}^{2n} \frac{1}{j} \\ &= \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} \\ &= \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{2}{2n+2} \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} \geq 0. \end{aligned}$$

To show that it is bounded above, we note that

$$a_n = \sum_{j=n+1}^{2n} \frac{1}{j} \leq \sum_{j=n+1}^{2n} \frac{1}{n} \leq 1.$$

5, C

4. (a) (i) TRUE. Let $x \in A \cup B$. Then, either $x \in A$ or $x \in B$. Without loss of generality we assume the first is true. Then there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset A \subset A \cup B$. Since x is arbitrary, it follows that $A \cup B$ is open.

2, A

- (ii) FALSE. $f(x) = x \sin(1/x)$, for $x \neq 0$ and $f(0) = 0$.

2, A

- (iii) TRUE. By the definition of limit there is $\delta > 0$ such that $f(x) \in (L - 1, L + 1)$ for $x \in (a - \delta, a + \delta) \setminus \{a\}$. Therefore $|f(x)| \leq \max\{|L + 1|, |L - 1|, |f(a)|\}$, for $x \in (a - \delta, a + \delta)$.

2, A

- (iv) FALSE. $f(x) = 0$, for all x and $g(x) = 1$ for x rational and $g(x) = 0$ for x irrational.

2, A

- (b) For all $L \in \mathbb{R}$, there exists an $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists $x \in (a - \delta, a + \delta) \setminus \{a\}$ such that $|f(x) - L| > \varepsilon_0$.

2, A

- (c) First note that, for all $x \neq 3$, we have

$$\left| \frac{7}{x^2 - 9} - 1 \right| = \left| \frac{7 - x^2 + 9}{x^2 - 9} \right| = \left| \frac{16 - x^2}{x^2 - 9} \right| = |4 - x| |4 + x| \left| \frac{1}{x - 3} \right| \left| \frac{1}{x + 3} \right|.$$

Assuming $3.5 < x < 5$, we get the inequalities $\frac{1}{|x-3|} < 2$, $\frac{1}{|x+3|} < 1$, $|4 + x| < 9$. In other words, if $|x - 4| < 1/2$ we obtain

$$\left| \frac{7}{x^2 - 9} - 1 \right| < 18|x - 4|.$$

It follows that, given $\varepsilon > 0$, if $\delta = \min(\varepsilon/18, 1/2)$, then $|x - 4| < \delta$ implies $\left| \frac{7}{x^2 - 9} - 1 \right| < \varepsilon$, as we wanted. (Partial credit allowed, e.g. if there are minor computational mistakes, or if the definition of limit is stated and/or used correctly).

5, B

- (d) Let $L = \sup\{f(x) : x < x_0\}$. Note that this supremum exists because this is a non-empty set and $f(x_0)$ is an upper bound (1 mark for justifying this). We need to show $\lim_{x \rightarrow x_0^-} f(x) = L$. To see this, fix $\varepsilon > 0$ and let $y < x_0$ be such that $L - \varepsilon < f(y) \leq L$, which exists by the definition of L (2 marks for using the definition supremum correctly, partial credit allowed). In particular, $0 \leq L - f(y) < \varepsilon$. Now let $\delta = |x_0 - y|$. Then for all x satisfying $x_0 - \delta < x < x_0$ we have $y < x$ which implies $|f(x) - L| = L - f(x) \leq L - f(y) < \varepsilon$. Which is what we wanted to show. (2 marks for the rest, partial credit allowed).

5, D

5. (a) (i) FALSE (1 mark). $f(x) = e^x$ (1 mark). 2, A
- (ii) FALSE (1 mark). $f(0) = 0$ and $f(x) = \sin(1/x)$ for $x \neq 0$ (1 mark). 2, A
- (iii) FALSE (1 mark). $f(x) = 0$ for $-1 \leq x < 1$ and $f(1) = 1$ (1 mark). 2, A
- (iv) FALSE (1 mark). $f(-1) = 1$, $f(1) = 2$ and $f(x) = 0$ for all $|x| < 1$ (1 mark). 2, A
- (b) $f''(x)$ exists if, first, there is $\delta > 0$ such that f is differentiable everywhere on $(x - \delta, x + \delta)$ (3 marks) and, second, $t \mapsto f'(t)$ is differentiable at x (2 marks). (Partial credit allowed) 5, B
- (c) If we differentiate the expression $f(x) = f(-x)$, by the Chain rule (2 mark, used and mentioned), we obtain $f'(x) = -f'(-x)$. Evaluating it at $x = 0$, we obtain $f'(0) = -f'(0)$, which implies $f'(0) = 0$ (1 mark for rest). 3, C
- (d) Let us proceed by contradiction. Assume, $f'(x) \neq 0$ for all $x \in \mathbb{R}$. By the Intermediate Value Property for derivatives (1 mark for mentioning), this implies that $f' > 0$ everywhere or $f' < 0$ everywhere. Without loss of generality we can assume the first case is true. By the Mean Value Theorem, we have that f is strictly monotone increasing (1 mark). This implies, for all $x > 1$, we have $f(-x) < f(-1) < f(1) < f(x)$. Taking $\lim_{x \rightarrow \infty}$ on this expression we conclude $L \leq f(-1) < f(1) \leq L$, which is a contradiction (4 marks for rest). (Partial marking allowed. Note there are several possible approaches to this question, and the marking scheme should be adjusted accordingly). 4, D

6. (a) (i) TRUE (1 mark). Consider the partition $P = \{a, b\}$. Then $L(f, P) = U(f, P)$ simplifies to $\sup_{[a,b]} f = \inf_{[a,b]} f$, which is equivalent to f being constant. (1 mark for rest) 2, A
- (ii) FALSE (1 mark). $f(x) = 0$ when $x \neq 0$ and $f(0) = 1$. (1 mark for rest) 2, A
- (iii) TRUE (1 mark). We showed in the lectures that every convex function is continuous on the interior of its domain (in this case $\mathbb{R}$) and that every continuous function is integrable. (1 mark for rest) 2, A
- (iv) TRUE (1 mark). The result is trivial if $x \in P$. Otherwise, $L(f, P') \geq L(f, P)$ simplifies to $\inf_{[t_i, p]} f \cdot (p - t_i) + \inf_{[p, t_{i+1}]} f \cdot (t_{i+1} - p) \geq \inf_{[t_i, t_{i+1}]} f \cdot (t_{i+1} - t_i)$, which follows from the inequalities $\inf_{[t_i, p]} f \geq \inf_{[t_i, t_{i+1}]} f$ and $\inf_{[p, t_{i+1}]} f \geq \inf_{[t_i, t_{i+1}]} f$. (1 mark for rest) 2, A
- (b) We proved during the lectures that the sum of integrable functions is integrable. Since f is integrable, it is enough to show that $h = g - f$ is integrable. Note that $h \geq 0$, in fact $h(x) = 1$ if $x = 0$ and $h(x) = 0$ otherwise. Given $\varepsilon > 0$, let $P = \{-1, -\varepsilon/2, \varepsilon/2, 1\}$. Then $U(h, P) - L(h, P) = \varepsilon$. Since $\varepsilon > 0$ is arbitrary it follows by the criterion of integrability proved in class, that h is integrable. (Partial marks allowed for mentioning the right theorems (1 mark) and for the right computation using a simple partial sum (3 marks)). 4, C
- (c) By the additivity of the integral with respect to the domain, we know that f is integrable on $[-1, 1]$ if and only if, it is integrable in both $[-1, 0]$ and $[0, 1]$. Moreover, in that case $\int_{-1}^1 f(x)dx = \int_{-1}^0 f(x)dx + \int_0^1 f(x)dx$. To conclude the integrability of the first integral, remember we showed in the lectures that changing the value of an integrable function at one point, preserves integrability as well as the value of the integral. Therefore $\int_{-1}^0 f(x)dx = \int_{-1}^0 x + 1dx$ as long as one of the integrals exist. Since $x + 1$ and $3 - 2x$ are continuous functions, and we proved all continuous functions are integrable, we conclude f is integrable. We can compute the value of the integral using the fundamental theorem of calculus. From $\frac{d}{dx}(\frac{x^2}{2} + x) = x + 1$ and $\frac{d}{dx}(3x - x^2) = 3 - 2x$ we obtain $\int_{-1}^0 x + 1 = -\frac{1}{2} + 1 = 1/2$ and $\int_0^1 3 - 2x = 2$. It follows that $\int_{-1}^1 f(x)dx = 5/2$. 5, B
- (d) If f is constant equal to zero, then its lower and upper sums with respect to any partition are zero. This implies its integral is zero.
- For the other direction, we argue by contradiction. Assume there exist $x \in [-1, 1]$ such that $f(x) > 0$. By continuity, we can assume WLOG that $x \in (-1, 1)$. Moreover, by continuity, there exist $\delta > 0$ such that $[x - \delta, x + \delta] \subset (-1, 1)$ and $f > f(x)/2$ on this set. Therefore, since $f \geq 0$, we have by the additivity of the integral with respect to the domain and the monotonicity of the integral:

$$\begin{aligned}
 \int_{-1}^1 f(x)dx &= \int_{-1}^{x-\delta} f(t)dt + \int_{x-\delta}^{x+\delta} f(t)dt + \int_{x+\delta}^1 f(t)dt \\
 &\geq \int_{x-\delta}^{x+\delta} f(t)dt \\
 &\geq \frac{f(x)}{2} 2\delta = f(x)\delta > 0.
 \end{aligned}$$

3, D

Review of mark distribution:

Total A marks: 48 of 48 marks

Total B marks: 35 of 30 marks

Total C marks: 18 of 18 marks

Total D marks: 24 of 24 marks

Total marks: 125 of 120 marks

MATH40002 Analysis I markers comments

Question 1

On 1a) This was a countability problem and overall performance was higher on this sort of question than it was in past years. Over 80% of students identified the set was uncountable and that Cantor's diagonal argument was a good way to show that (although not the only way). The most common mistake in running the diagonal argument was that it was not clear that the new sequence constructed by students did in fact converge. A common issue here is that, even if one knows that, for each $j \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_{j,n} = 0$, this does not mean $\lim_{n \rightarrow \infty} a_{n,n} = 0$.

On 1b) Most students got this, but many students tried to directly use comparison test which technically does not apply here (you need to reindex the subsequence and add in 0's to make this work).

On 1c) Many students assumed that a_n is bounded, which may not be the case. However, a_n certainly has a bounded subsequence.

On 1d) - not many students got this, but this problem was categorized as high difficulty so that wasn't too unexpected. The idea behind this problem is similar to what we saw with rearrangements, but in this problem you aren't actually finding a rearrangement. A fair number of students started trying to construct a rearrangement which is different than constructing a subsequence of partial sums. Overall, performance on this question was higher than my expectations - so good job!

Question 2

To get a reasonable amount of marks on this question it was merely necessary to know the proper definitions of properties such as Cauchy, Bounded, Convergence etc... In order to get full marks it is then encouraged that students should learn how to properly manipulate epsilon convergence proofs in a precise manner (for instance getting precisely that for all ϵ there exists an N such that for all $N > N$ $|a_n - b_n| < \epsilon$, NOT $|a_n - b_n| < f(\epsilon)$ where f is a function that you hand-wave).

- Question 3
- Q3 a) i) This part was done well by most people, who were awarded full marks. The most common mistakes were either assuming that $(\sum a_n)^2 = (\sum a_n^2)$, or using the ratio test incorrectly.
- a) ii) This part was the one that proved hardest for the students. Very few people got full marks, and a lot of people did not attempt. The most common mistake was working under the false assumption that a_n was a positive sequence, or obtaining wrong bounds for the tail of the sum.
- b) was also done really well. Most people understood the task and picked the right N . The most common mistake was arguing that $a_{f(n)}$ was a subsequence of a_n , which it wasn't, and using that bijections on N are monotonic.
- c) was also mostly done well. Students who understood they needed to prove that the sequence was monotone and bounded normally had no issues doing so and got full marks. The common mistakes appeared when students tried to find the limit explicitly, or showing that the sequence was Cauchy.
- Question 4
- Overall, students made a good effort to show examples and explain their partial reasoning. An important mistake that is worth mentioning was regarding part (d). Several people started their argument by assuming that the limit exists, this is not a given and is the central issue of the question. The hint we are giving you is that the limit should be equal to $S(x_0)$, knowing this helps with proving that the limit exists.

Question 5

In general there was a very wide range of performances on this question, from significant few achieving high or close to full marks to others achieving very few marks and many achieving marks in between these.

(5a) The wide range of performances on question 5 was reflected in part (5A).

(5ai) Several students answered this part correctly and were able to find valid and good counterexamples to support their answers. Those who did not answer correctly did not realise that the (only) minimum of the function could very well be attained at either of the endpoints.

(5aii) Again, several students answered this part correctly and were able to find valid counterexamples to support their answers, although a few needed to be more careful when giving counterexamples, accounting for points in the domain on which the 'general' functions are not defined. Many of those who answered (ii) incorrectly seemed to believe that the IVP was in some way equivalent with continuity.

(5aiii) Much fewer students answered this part correctly. Those who did answer correctly largely (though not always) gave correct counterexamples. The main reason why students did not answer this part correctly was that they did not remember that convex functions on a CLOSED interval (as is the case with $[-1,1]$) may fail to be continuous at the endpoints. Instead, they believed that convexity on a closed interval automatically implies continuity on that interval.

(5aiv) Many students did not answer this part correctly. By far the most common error was to try to prove that the statement given was true by use of the Mean Value Theorem, which does not apply here as the function f is not necessarily continuous at the endpoints.

(5b) Several students were able to obtain some marks for this question, and many did understand that it was necessary to first ensure (as a prerequisite) that the function needed to be differentiable at x . What a considerable number of students failed to state is that f needs to be differentiable in some open interval containing x , which precluded them from obtaining full marks. A smaller number of students seemed to have misread the question, thinking that it was asking what it meant for f'' rather than $f''(x)$ to exist.

(5c) Most students obtained some credit for this question though notably fewer obtained full credit. There were several different methods used to try to answer this question. Some used the chain rule (though a few did not spell out explicitly their use thereof), whilst others used the local comparison test. Among many of the students who did not obtain full marks, most tried to use Rolle's theorem or the mean value theorem and that $f(c)=f(-c)$ to obtain that there exists $x \in (-c,c)$ such that $f'(c)=0$. However, they could not fully or correctly explain why this implies that $f'(0)=0$.

(5d) Very many students knew the general idea of how to prove the statement in the question, and therefore obtained some (if sometimes little) credit. Again, different methods were used, with some using proof by contradiction, and some attempting to reduce the problem to an application of Rolle's theorem/ the mean value theorem. Almost all students did identify that if the function is constant, then any point is a critical point. Much fewer than half of the students obtained

full marks, and these students gave clear, detailed, rigorous proofs. Much more obtained some credit and it was clear that they had grasped the general idea of how to answer the question. However, their answers contained many (or significant omissions), such as why one can find two points a and b such that $f(a)=f(b)$ to apply Rolle's theorem, or why one can choose a and b so that any maximum/ minimum is not attained at either. Some students omitted key explanations in their answers, such as why the absence of a critical point implies monotonicity of the function (e.g. failing to mention the IVT for derivatives).

Question 6

Performance in question 6 was very varied.

(a) This part of the question, when attempted, was generally answered correctly. Some students tried to show part (ii) to be true and then use this in part (d), which does not work. Some arguments in other parts lacked the sufficient rigour expected when arguing about integration in an analysis course.

(b) Some students found the simpler proof of this result that consists in writing g as $f + h$ and showing the integrability of h . The majority of students, however, tried to prove directly that g is integrable. There are some subtle issues here that many students have not noted in their solution, e.g., the need to pick specific partitions to get the desired relation between lower and upper sums. Many students also wrongly assumed that the lower sums of f and g had to coincide, and that the maximum in the interval containing 0 for g was $g(0)$. More generally, many students assumed that f was continuous, which is not part of the assumptions of the problem. This question possibly best reflects the varying range of performances for 6.

(c) Most students computed the integration correctly (up to some small arithmetical mistakes), but there was a significant number of responses lacking a proof that f is integrable.

(d) Many students forgot to prove the right-to-left direction or simply claimed it was trivial. However, a formal argument was expected -- noting that lower and upper sums had to be all zero was enough. A number of students also attempted to prove the converse without using the assumption of continuity, which then fails.