

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2026

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Analysis 1

Date: Tuesday, 05 May, 2026

Time: Start time 14:00 – End time 16:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) For $n \geq 1$ define

$$a_n := \sqrt{n+1} - \sqrt{n}.$$

- (i) Show that $a_n \rightarrow 0$. (4 marks)
(ii) Prove that $\sqrt{n} a_n \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$, using the ε - N definition. (2 marks)
(b) Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Justify your answer. (3 marks)
(c) Show that the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges. Moreover, if s is its sum and $s_N := \sum_{n=1}^N (-1)^{n-1} a_n$, prove that

$$|s - s_N| \leq a_{N+1} \quad (N \geq 1).$$

(5 marks)

- (d) Consider the power series (complex variable z)

$$F(z) := \sum_{n=1}^{\infty} a_n z^n.$$

- (i) Show that $F(z)$ converges absolutely for all $|z| < 1$, and that it diverges for $z = 1$. Deduce that the radius of convergence is $R = 1$. (3 marks)
(ii) Investigate convergence on the unit circle at the points

$$z = -1, \quad z = e^{i\theta} \text{ with } \theta \not\equiv 0 \pmod{2\pi}.$$

(3 marks)

(Total: 20 marks)

2. (a) Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous. Assume that $f(q) = 0$ for every rational $q \in [0, 1]$. Prove that $f(x) = 0$ for every $x \in [0, 1]$. (6 marks)
(b) Let $f : (0, 1) \rightarrow \mathbb{R}$ be continuous. Assume that the one-sided limits

$$\lim_{x \rightarrow 0^+} f(x) \quad \text{and} \quad \lim_{x \rightarrow 1^-} f(x)$$

exist and are finite.

- (i) Define a function $F : [0, 1] \rightarrow \mathbb{R}$ by setting $F(x) = f(x)$ for $x \in (0, 1)$ and choosing $F(0), F(1)$ appropriately. Prove that F is continuous on $[0, 1]$. (4 marks)
(ii) Using Heine's theorem, deduce that f is uniformly continuous on $(0, 1)$. (4 marks)
(c) Consider the function $g : (0, 1) \rightarrow \mathbb{R}$ given by

$$g(x) = \sin\left(\frac{1}{1-x}\right).$$

- (i) Show that g is continuous and bounded on $(0, 1)$. (2 marks)
(ii) Show that g is *not* uniformly continuous on $(0, 1)$. (4 marks)

(Total: 20 marks)

3. (a) For the following statements indicate whether they are true or false (1 mark), and give a short proof or counterexample of your answer.
- (i) Let $g : [0, 1] \rightarrow [0, 1]$ and $f : [0, 1] \rightarrow \mathbb{R}$ be Darboux integrable. Then the function $f \circ g$ is Darboux integrable on $[0, 1]$. (3 marks)
 - (ii) If f, g are convex on an interval, then the function $\max\{f, g\}$ is convex on that interval. (3 marks)
 - (iii) Let $f : I \rightarrow J$ be a bijection between the open intervals I, J with inverse $g : J \rightarrow I$. Assume both f and g are differentiable on their respective domains. Then $f'(x) \neq 0$ for all $x \in I$. (3 marks)
 - (iv) The function f defined as $f(x) = x^2$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise, is twice differentiable at $x = 0$. (3 marks)
- (b) For the following definitions and theorems, state them precisely emphasising all the correct hypotheses.
- (i) State the definition of a Darboux integrable function (you must define upper and lower sums). (2 marks)
 - (ii) State Taylor's Theorem for order n with Integral form of the remainder. (3 marks)
 - (iii) State L'Hopital's Rule (in either of the forms we saw during the lectures or problem sheets). (3 marks)

(Total: 20 marks)

4. (a) Show that if $f : [a, b] \rightarrow \mathbb{R}$ is bounded and continuous except at a finite number of points, then f is Darboux integrable on $[a, b]$. (5 marks)
- (b) Let $f : [0, \infty) \rightarrow \mathbb{R}$ be continuous and twice differentiable on $(0, \infty)$. Assume that $f''(x) \geq 0$ for all $x > 0$ and that f is bounded on $[0, \infty)$.
- (i) Show that f is non-increasing on $(0, \infty)$. (2 marks)
 - (ii) Show from the definitions of infimum and limit, that $\lim_{x \rightarrow \infty} f(x)$ exists and is finite. (3 marks)
 - (iii) Show that $\lim_{x \rightarrow \infty} f'(x) = 0$. (4 marks)
 - (iv) Show that $xf'(x) \leq f(2x) - f(x) \leq xf'(2x)$ for all $x > 0$. (4 marks)
Hint: integrate $f'(x)$ on the interval $[x, 2x]$.
 - (v) Conclude from (iv) that $\lim_{x \rightarrow \infty} xf'(x) = 0$. (2 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2026

This paper is also taken for the relevant examination for the Associateship.

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XXX (Solutions)

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1. (a) (i) Since $\sqrt{n+1} > \sqrt{n}$ we have $a_n > 0$. Rationalising,

meth seen ↓

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Hence for all n ,

$$0 < a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}}.$$

Let $\varepsilon > 0$. Choose $N > \varepsilon^{-2}$. Then for $n \geq N$,

$$0 < a_n \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon.$$

Thus $a_n \rightarrow 0$.

4, A

(ii) From (i),

$$\sqrt{n} a_n = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}.$$

Let $\varepsilon > 0$. Since $\frac{1}{n} \rightarrow 0$, we may choose N such that for all $n \geq N$,

$$\left| \sqrt{1 + \frac{1}{n}} - 1 \right| < \varepsilon.$$

Then for $n \geq N$ we have

$$\left| \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} - \frac{1}{2} \right| = \frac{\left| \sqrt{1 + \frac{1}{n}} - 1 \right|}{2(\sqrt{1 + \frac{1}{n}} + 1)} \leq \frac{1}{2} \left| \sqrt{1 + \frac{1}{n}} - 1 \right| < \frac{\varepsilon}{2}.$$

So $\sqrt{n} a_n \rightarrow \frac{1}{2}$.

2, B

- (b) The partial sums telescope:

meth seen ↓

$$\sum_{n=1}^N a_n = \sum_{n=1}^N (\sqrt{n+1} - \sqrt{n}) = \sqrt{N+1} - 1.$$

Since $\sqrt{N+1} - 1 \rightarrow \infty$ as $N \rightarrow \infty$, the series $\sum_{n=1}^{\infty} a_n$ diverges (to $+\infty$).

3, A

- (c) From (a)(i) we have $a_n > 0$ and $a_n \rightarrow 0$. Also a_n is decreasing since

meth seen ↓

$$a_{n+1} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}} < \frac{1}{\sqrt{n+1} + \sqrt{n}} = a_n.$$

Hence by the alternating series test, $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

For the remainder estimate: set $s = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ and $s_N = \sum_{n=1}^N (-1)^{n-1} a_n$.

By the alternating series remainder theorem (or by bounding tails),

$$|s - s_N| \leq a_{N+1} \quad (N \geq 1).$$

5, B

- (d) (i) If $|z| < 1$, then using $a_n \leq \frac{1}{\sqrt{n}}$ from (a)(i),

meth seen ↓

$$\sum_{n=1}^{\infty} |a_n z^n| \leq \sum_{n=1}^{\infty} |z|^n,$$

which converges. Hence $F(z)$ converges absolutely for all $|z| < 1$.

At $z = 1$ we have $F(1) = \sum_{n=1}^{\infty} a_n$, which diverges by part (b). Therefore the radius of convergence is $R = 1$.

3, C

(ii) At $z = -1$ we have $F(-1) = \sum_{n=1}^{\infty} (-1)^n a_n$, which converges by part (c) (conditional, since $\sum a_n$ diverges).

Let $z = e^{i\theta}$ with $\theta \not\equiv 0 \pmod{2\pi}$. Then the partial sums of $\sum_{n=1}^N z^n$ are bounded:

$$\sum_{n=1}^N z^n = z \frac{1 - z^N}{1 - z}, \quad \left| \sum_{n=1}^N z^n \right| \leq \frac{2}{|1 - z|}.$$

Since a_n is decreasing and $a_n \rightarrow 0$, Dirichlet's test implies that $\sum_{n=1}^{\infty} a_n z^n$ converges for all such z . The convergence is not absolute on $|z| = 1$ because $|a_n z^n| = a_n$ and $\sum a_n$ diverges.

3, C

2. (a) Fix $x \in [0, 1]$. Choose rationals $q_n \in [0, 1] \cap \mathbb{Q}$ with $q_n \rightarrow x$ (density of $\mathbb{Q}$). By continuity,

meth seen ↓

$$f(x) = \lim_{n \rightarrow \infty} f(q_n) = \lim_{n \rightarrow \infty} 0 = 0.$$

Since x was arbitrary, $f \equiv 0$ on $[0, 1]$.

6, A

- (b) (i) Define

meth seen ↓

$$F(x) = f(x) \text{ for } x \in (0, 1), \quad F(0) := \lim_{x \rightarrow 0^+} f(x), \quad F(1) := \lim_{x \rightarrow 1^-} f(x).$$

Then F is continuous on $(0, 1)$ because it equals f there. To check continuity at 0, let $\varepsilon > 0$. By definition of $F(0)$ there exists $\delta > 0$ such that $|f(x) - F(0)| < \varepsilon$ whenever $0 < x < \delta$. But for such x , $F(x) = f(x)$, hence $|F(x) - F(0)| < \varepsilon$ for $0 < x < \delta$, so F is continuous at 0. The argument at 1 is identical using the limit $\lim_{x \rightarrow 1^-} f(x) = F(1)$. Thus F is continuous on $[0, 1]$.

4, B

(ii) By Heine–Cantor (Heine’s theorem), a continuous function on a compact interval is uniformly continuous. Since F is continuous on $[0, 1]$, it is uniformly continuous on $[0, 1]$. Restricting to $(0, 1)$, we get: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in (0, 1)$, if $|x - y| < \delta$ then $|f(x) - f(y)| = |F(x) - F(y)| < \varepsilon$. Hence f is uniformly continuous on $(0, 1)$.

4, D

- (c) (i) The map $x \mapsto 1 - x$ is continuous on $(0, 1)$ and never zero there, hence $x \mapsto \frac{1}{1-x}$ is continuous on $(0, 1)$. Since $\sin$ is continuous on $\mathbb{R}$, their composition $g(x) = \sin\left(\frac{1}{1-x}\right)$ is continuous on $(0, 1)$. Also $|\sin(\cdot)| \leq 1$, so $|g(x)| \leq 1$ for all $x \in (0, 1)$; thus g is bounded.

2, A

(ii) Let

$$y_n := 1 - \frac{1}{2\pi n}, \quad x_n := 1 - \frac{1}{2\pi n + \frac{\pi}{2}} \quad (n \geq 1).$$

Then $x_n, y_n \in (0, 1)$ and $x_n \rightarrow 1^-$, $y_n \rightarrow 1^-$. Moreover,

$$g(y_n) = \sin(2\pi n) = 0, \quad g(x_n) = \sin\left(2\pi n + \frac{\pi}{2}\right) = 1,$$

so $|g(x_n) - g(y_n)| = 1$ for all n . On the other hand,

$$|x_n - y_n| = \left| \frac{1}{2\pi n} - \frac{1}{2\pi n + \frac{\pi}{2}} \right| \rightarrow 0.$$

Take $\varepsilon = \frac{1}{2}$. For any $\delta > 0$, choose n large enough that $|x_n - y_n| < \delta$; then $|g(x_n) - g(y_n)| = 1 \geq \varepsilon$. Therefore g is not uniformly continuous on $(0, 1)$.

4, D

3. (a) **(i) False.** Take g to be Thomae's function on $[0, 1]$ (Darboux integrable) and f defined as $f(0) = 1$ and $f(x) = 0$ for $x \neq 0$ on $[0, 1]$ (Darboux integrable). Then $(f \circ g)(x) = 1$ iff $x \notin \mathbb{Q}$, and zero otherwise. This is the Dirichlet function which is not Darboux integrable.

3, D

(ii) True. Let $h = \max\{f, g\}$. For x, y and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \leq \lambda h(x) + (1 - \lambda)h(y),$$

and similarly for g . Taking the maximum gives $h(\lambda x + (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y)$, so h is convex.

3, B

(iii) True. Since $g = f^{-1}$, we have $g \circ f = \text{id}_I$. Differentiating,

$$1 = (g \circ f)'(x) = g'(f(x)) f'(x),$$

so $f'(x) \neq 0$ for all $x \in I$.

3, A

(iv) False. The function is not differentiable on a neighborhood of 0. In particular $F'(x)$ is not defined near 0, therefore it cannot be twice differentiable at 0.

3, A

- (b) **(i) Definition (Darboux integrable).** Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded and let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition. Set

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x),$$

and define the upper and lower sums

$$U(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1}), \quad L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1}).$$

Define $\alpha := \inf_P U(f, P)$ and $\beta := \sup_P L(f, P)$. Then f is *Darboux integrable* on $[a, b]$ if $\alpha = \beta$.

2, C

(ii) Taylor's Theorem (order n , integral remainder). Assume f has $n + 1$ continuous derivatives on an interval containing a and x . Then

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k + \frac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t) dt.$$

3, A

(iii) L'Hôpital's Rule (one standard form). Let f, g be differentiable on (a, b) with $g'(x) \neq 0$ on (a, b) and assume $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x) = 0$. If the limit $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L$ exists, then

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L.$$

3, B

4. (a) Assume $f : [a, b] \rightarrow \mathbb{R}$ is bounded and continuous except at finitely many points $c_1, \dots, c_m$. Fix $\varepsilon > 0$ and let $M = \sup_{[a, b]} |f|$. Choose disjoint intervals I_j around c_j so that the total length satisfies $\sum_{j=1}^m |I_j| < \varepsilon/(4M)$. On the compact set $K = [a, b] \setminus \bigcup_{j=1}^m I_j$, the function f is continuous, hence uniformly continuous, so there is a partition P fine enough so that the oscillation of f on each subinterval contained in K is $< \varepsilon/(2(b-a))$. Finally, if we understand P as a partition of $[a, b]$, then

$$U(f, P) - L(f, P) \leq \frac{\varepsilon}{2(b-a)}(b-a) + 2M \sum_{j=1}^m |I_j| < \frac{\varepsilon}{2} + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon.$$

Hence f is Darboux integrable.

5, D

- (b) (i) Since $f''(x) \geq 0$ on $(0, \infty)$, the derivative f' is non-decreasing on $(0, \infty)$. If f were not non-increasing, there would exist $0 < x < y$ with $f(y) > f(x)$. By the Mean Value Theorem, there is $c \in (x, y)$ such that $f'(c) = \frac{f(y) - f(x)}{y - x} > 0$. As f' is non-decreasing, $f'(t) \geq f'(c) > 0$ for all $t \geq c$, hence for $z > c$,

$$f(z) = f(c) + \int_c^z f'(t) dt \geq f(c) + f'(c)(z - c) \xrightarrow{z \rightarrow \infty} +\infty,$$

contradicting that f is bounded above. Thus f is non-increasing on $(0, \infty)$, and by continuity at 0 it is non-increasing on $[0, \infty)$.

2, A

(ii) Since f is non-increasing and bounded below, the number $\ell := \inf_{x \geq 0} f(x)$ is finite. Let $\varepsilon > 0$. By definition of infimum, choose $x_\varepsilon \geq 0$ with $f(x_\varepsilon) < \ell + \varepsilon$. For all $x \geq x_\varepsilon$, monotonicity gives $\ell \leq f(x) \leq f(x_\varepsilon) < \ell + \varepsilon$, hence $|f(x) - \ell| < \varepsilon$. Therefore $\lim_{x \rightarrow \infty} f(x) = \ell$ exists and is finite.

3, B

(iii) From (i), f is non-increasing, so for $x > 0$ and $h > 0$,

$$\frac{f(x+h) - f(x)}{h} \leq 0,$$

and letting $h \rightarrow 0^+$ gives $f'(x) \leq 0$ for all $x > 0$. Also $f'' \geq 0$ implies f' is non-decreasing, hence f' is monotone and bounded above by 0, so $L := \lim_{x \rightarrow \infty} f'(x)$ exists with $L \leq 0$. If $L < 0$, then for some $R > 0$ we have $f'(x) \leq L/2 < 0$ for all $x \geq R$, and

$$f(x) = f(R) + \int_R^x f'(t) dt \leq f(R) + \frac{L}{2}(x - R) \xrightarrow{x \rightarrow \infty} -\infty,$$

contradicting boundedness below. Hence $L = 0$, i.e. $\lim_{x \rightarrow \infty} f'(x) = 0$.

4, C

(iv) Since f' is non-decreasing, for $t \in [x, 2x]$ we have $f'(x) \leq f'(t) \leq f'(2x)$. Integrating over $[x, 2x]$ yields

$$xf'(x) \leq \int_x^{2x} f'(t) dt \leq xf'(2x).$$

By the Fundamental Theorem of Calculus, $\int_x^{2x} f'(t) dt = f(2x) - f(x)$, hence

$$xf'(x) \leq f(2x) - f(x) \leq xf'(2x) \quad (x > 0).$$

(v) Apply (iv) at x and at $x/2$ to obtain

$$xf'(x) \leq f(2x) - f(x), \quad 2(f(x) - f(x/2)) \leq xf'(x).$$

Thus

$$2(f(x) - f(x/2)) \leq xf'(x) \leq f(2x) - f(x).$$

Let $\ell = \lim_{x \rightarrow \infty} f(x)$ from (ii). Then $f(2x) - f(x) \rightarrow 0$ and $f(x) - f(x/2) \rightarrow 0$, so by the squeeze theorem $\lim_{x \rightarrow \infty} xf'(x) = 0$.

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total Mastery marks: 0 of 20 marks

Total marks: 80 of 80 marks