

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Analysis 1

Date: Thursday, 6 May 2021

Time: 09:00 to 12:00

Time Allowed: 3 hours

Upload Time Allowed: 45 minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

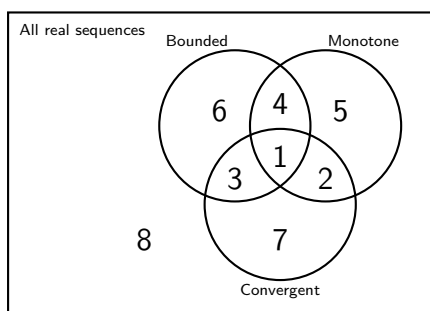
Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Unless explicitly stated, you don't need to give any proofs for the parts of this first problem.

- (a) Below is a Venn Diagram of all real sequences with three regions corresponding to bounded, monotone, and convergent sequences. List all the numbered regions in this Venn Diagram that must be empty, that is, every region where it would be impossible to find any sequences. (5 marks)



- (b) Let $A \subset \mathbb{N}$ and let $\mathcal{P}(A)$ be the power set of A , that is, the collection of all subsets of A .
- (i) True/False: It is possible for $\mathcal{P}(A)$ to be countably infinite. (1 mark)
- (ii) True/False: It is possible for $\mathcal{P}(A)$ to be uncountably infinite. (1 mark)
- (c) Given a sequence $(a_n)_{n=1}^{\infty}$, suppose that we define

$$S = \left\{ a \in \mathbb{R} : \text{there is a rearrangement of } \sum_{n=1}^{\infty} a_n \text{ converging to } a \right\}.$$

- (i) True/False: It is possible for S to be empty. (1 mark)
- (ii) True/False: It is possible for S to be countably infinite. (1 mark)
- (iii) True/False: It is possible for S to be uncountably infinite. (1 mark)
- (d) (i) Find a real sequence $(a_n)_{n=1}^{\infty}$ without any convergent subsequence. (1 mark)
- (ii) Find a real sequence $(a_n)_{n=1}^{\infty}$ that is unbounded but has a convergent subsequence. (1 mark)
- (iii) True/False: Suppose the sequence $(a_n)_{n=1}^{\infty}$ satisfies $a_{n+1} - a_n \rightarrow 0$. Then it must be the case that $(a_n)_{n=1}^{\infty}$ converges. (1 mark)
- (iv) True/False: Suppose the sequence $(a_n)_{n=1}^{\infty}$ consists of positive terms and satisfies $a_{n+1}/a_n \rightarrow 1/2$. Then it must be the case that $(a_n)_{n=1}^{\infty}$ converges. (1 mark).
- (e) Suppose that the power series $\sum_{n=1}^{\infty} a_n z^n$ has a radius of convergence $R_1 \in (0, \infty)$ and the power series $\sum_{n=1}^{\infty} b_n z^n$ has a radius of convergence $R_2 \in (0, \infty)$. Let R be the radius of convergence of $\sum_{n=1}^{\infty} a_n b_n z^n$, carefully prove that $R \geq R_1 R_2$. (6 marks)

(Total: 20 marks)

2. (a) Prove the following statements **using first principles**, that is, using **only** the **definition** of convergence and describing fairly explicitly how one can choose $N \in \mathbb{N}$ for each given $\epsilon > 0$.
- (i) The sequence $(a_n)_{n=1}^{\infty}$ given by $a_n = \frac{1}{n^2 - n + 1}$ converges to 0. (3 marks)
- (ii) If one has $a_n \rightarrow a$, then prove that $a_n^3 \rightarrow a^3$. (3 marks)
- (b) Find an example of a divergent sequence $(a_n)_{n=1}^{\infty}$ such that, the sequence $(b_n)_{n=1}^{\infty}$ defined by setting $b_n = \frac{1}{n^2} \sum_{j=1}^n j \cdot a_j$ converges. Carefully prove your answer (namely, that your choice of $(a_n)_{n=1}^{\infty}$ diverges and $(b_n)_{n=1}^{\infty}$ converges). (6 marks)
- (c) Let $A \subset (0, \infty)$ be non-empty and bounded above and $B \subset (1, \infty)$ be non-empty. Prove that $\sup\{\frac{a}{b} : a \in A, b \in B\}$ is given by $\sup(A)/\inf(B)$. (5 marks)
- (d) Let $(a_n)_{n=1}^{\infty}$ be a sequence of non-negative terms, and let $(b_i)_{i=1}^{\infty}$ be given by $b_i = \sum_{j=1}^{2i} a_j$.
Supposing that $(b_i)_{i=1}^{\infty}$ converges, prove that $\sum_{n=1}^{\infty} a_n$ converges. (3 marks)

(Total: 20 marks)

3. (a) Prove the convergence of the sequence $(a_n)_{n=1}^{\infty}$ given by setting

$$a_n = \frac{1}{\sqrt{n(n+1)}} + \frac{1}{\sqrt{(n+1)(n+2)}} + \cdots + \frac{1}{\sqrt{(2n-1)2n}}$$

for each $n \in \mathbb{N}$. (6 marks)

- (b) Suppose that the sequence $(a_n)_{n=1}^{\infty}$ is unbounded. Prove that there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ with the property that $\lim_{k \rightarrow \infty} (1/a_{n_k}) = 0$. (5 marks)
- (c) Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be bounded sequences.
- (i) Prove that $\sup_{n \in \mathbb{N}} (a_n + b_n) \leq \left(\sup_{n \in \mathbb{N}} a_n \right) + \left(\sup_{n \in \mathbb{N}} b_n \right)$. (4 marks)
- (ii) Find an example where $\sup_{n \in \mathbb{N}} (a_n + b_n) < \left(\sup_{n \in \mathbb{N}} a_n \right) + \left(\sup_{n \in \mathbb{N}} b_n \right)$. (2 marks)
- (d) Suppose that $(a_n)_{n=1}^{\infty}$ is sequence of positive terms and that $\sum_{n=1}^{\infty} a_n$ converges. Define the sequence $(b_n)_{n=1}^{\infty}$ by setting $b_n = \frac{1}{n} \sum_{i=1}^n a_i$. Prove that $\sum_{n=1}^{\infty} b_n$ diverges. (3 marks)

(Total: 20 marks)

4. (a) (i) Prove that for all integers $k \geq 1$, the function

$$f(x) = \begin{cases} x^k \sin\left(\frac{1}{x^3}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$. (3 marks)

- (ii) Prove that $f(x)$ is differentiable at $x = 0$ for all $k \geq 2$, and compute $f'(0)$. (3 marks)

- (b) (i) Let $k > 0$ be a positive integer. Prove that there is a real number x such that $x^{2k} = 2e^x - 3$. (3 marks)

- (ii) Give (with proof) a non-constant polynomial $p(x)$ such that $p(x) = 2e^x - 3$ has no real solutions. (2 marks)

- (c) (i) (No proof necessary for this part.) For each of the following subsets of $\mathbb{R}$:

(1) $(-1, 1)$ (1 mark)

(2) $\{0\} \cup [2, \infty)$ (1 mark)

(3) $\mathbb{Q}$ (1 mark)

(4) $\{x \in \mathbb{R} \mid \sin(\pi x) = 0\}$ (1 mark)

(5) $\{x^2 \mid -1 < x < 1\}$ (1 mark)

state whether it is (A) open, (B) closed, (C) both, (D) neither.

- (ii) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, and let $U \subset \mathbb{R}$ be open. Prove that if $f'(x) > 0$ for all $x \in U$, then $f(U) \subset \mathbb{R}$ is open.

(4 marks)

(Total: 20 marks)

5. (a) (i) Give an example (no proof necessary) of a bounded, continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with no local minima or maxima. (2 marks)

- (ii) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ has infinitely many derivatives, and that for some $n \geq 2$ there are $n + 1$ real numbers $x_0 < x_1 < \dots < x_n$ such that $f(x_i) = 40002$ for each $i = 1, 2, \dots, n$. Prove that there is some $y \in \mathbb{R}$ such that $f^{(n)}(y) = 0$. (4 marks)

- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable. Use the mean value theorem to prove each of the following.

(i) If $f'(x) = c$ for all $x \in \mathbb{R}$, then $f(x) = cx + f(0)$ for all $x \in \mathbb{R}$. (2 marks)

(ii) If the equation $f(x) = 3x$ has two distinct solutions, then the equation $f'(x) = 3$ also has a solution. (3 marks)

(iii) If $x > 0$ then $\frac{x}{x^2+1} < \tan^{-1}(x) < x$. (3 marks)

- (c) Define $f : (-1, 1) \rightarrow \mathbb{R}$ by $f(x) = \frac{1}{1-x^3}$.

(i) Compute $\frac{1}{12!} \cdot f^{(12)}(0)$. (2 marks)

(ii) Prove for $0 < x < 1$ the inequality $\frac{f(x)+1}{2} \geq f\left(\frac{x}{2}\right)$. (4 marks)

(Total: 20 marks)

6. (a) Define $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = x^r$ for some real number $r \geq 0$. For each integer $n \geq 1$, define a partition $P_n = (0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1)$ of the interval $[0, 1]$.

- (i) Write an expression for each of the Darboux sums $L(f, P_n)$ and $U(f, P_n)$. (2 marks)
- (ii) Compute the difference $U(f, P_n) - L(f, P_n)$ in terms of n and r . (2 marks)
- (iii) Prove the inequality

$$0 \leq (1^r + 2^r + \dots + n^r) - \frac{n^{r+1}}{r+1} \leq n^r$$

for all integers $n \geq 1$. (2 marks)

(b) (i) Prove that the improper integral $\int_1^\infty e^{-x^3} dx$ exists. (3 marks)

(ii) Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous function satisfying $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = 1$. Prove that

$$\lim_{x \rightarrow \infty} \frac{1}{x^2} \int_0^x f(t) dt$$

exists and compute its value. (3 marks)

(c) Define $f : [0, \infty) \rightarrow \mathbb{R}$ by $f(x) = \int_0^x \sin(\sqrt{t^2 + 1}) dt$.

- (i) Determine all $x > 0$ at which f has a local minimum or local maximum, and state whether it's a minimum or a maximum. (4 marks)
- (ii) Prove that $\lim_{x \rightarrow \infty} f(x)$ does not exist. You may use the inequality

$$y > x > 0 \implies y - x > \sqrt{y^2 + 1} - \sqrt{x^2 + 1}$$

without proof. (4 marks)

(Total: 20 marks)

Solutions

1. (a) Region 2 - cannot be convergent but not bounded.
Region 4 - cannot be bounded and monotone but not convergent.
Region 7 - cannot be convergent but not bounded. (5 marks)
- (b) If A is finite then $\mathcal{P}(A)$ is finite, while if A is infinite, $\mathcal{P}(A)$ is uncountably infinite, therefore the answers are
- (i) False. (1 mark)
- (ii) True. (1 mark)
- (c) (i) True, for instance, take $a_n = \frac{1}{n}$. (1 mark)
- (ii) False, the series can either take on a finite number of values (either 0, or 1, depending on whether it is absolutely convergent or only has divergent rearrangements) or it takes every possible value (if it is conditionally convergent) (1 mark)
- (iii) True, take a conditionally convergent series can be arranged to converge to any real number. (1 mark)
- (d) (i) Can take $a_n = n$, in fact any monotone unbounded sequence would work. (1 mark)
- (ii) Can take $a_n = n$ for n odd and $a_n = 0$ for n even. Find a real sequence $(a_n)_{n=1}^{\infty}$ that is unbounded but has a convergent subsequence. (1 mark)
- (iii) False, can take $a_n = \sum_{j=1}^n \frac{1}{j}$. (1 mark)
- (iv) True, it must be the case that $a_n \rightarrow 0$, see Example 3.23 in the lecture notes. (1 mark).
- (e) Given a power series $\sum_{n=1}^{\infty} c_n z_n$ with radius of convergence R , we know from the proof of Theorem 4.35 of the lecture notes, then $c_n r^n \rightarrow 0$ implies that $r \leq R$.
Now let $0 \leq r < R_1 R_2$, then we can write $r = \ell R_1 R_2$ for some $\ell \in [0, 1)$. We then have $a_n b_n r^n = a_n (\ell^{1/2} R_1)^n \cdot b_n (\ell^{1/2} R_2)^n$. Since $\ell^{1/2} \in [0, 1)$, it follows that $a_n (\ell^{1/2} R_1)^n \rightarrow 0$ and $b_n (\ell^{1/2} R_2)^n \rightarrow 0$ and so by the algebra of limits, $a_n b_n r^n \rightarrow 0$.
So if R is the radius of convergence, $r < R_1 R_2 \Rightarrow r \leq R$, and so by Exercise 4.41 we have $R \geq R_1 R_2$. (6 marks)

(Total: 20 marks)

2. (a) (i) We first note that, for any $n \geq 2$, one has that $n^2 \geq 2n > 2(n-1)$ so

$$n \geq 2 \Rightarrow n^2 - n + 1 = n^2 - (n-1) \geq \frac{n^2}{2}.$$

Now, let $\epsilon > 0$, then fix $N = \lceil \sqrt{\frac{2}{\epsilon}} \rceil + 1$. Note that $N \geq 2$. We then have, for any $n \geq N$,

$$\left| \frac{1}{n^2 - n + 1} - 0 \right| \leq \left| \frac{2}{n^2} \right| \leq \frac{2}{N^2} < \epsilon.$$

Therefore, $\frac{1}{n^2 - n + 1} \rightarrow 0$. (3 marks)

- (ii) First note that, since $a_n \rightarrow a$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow |a - a_n| \leq 1$. In particular, for $n \geq N_1$, $|a_n| \leq |a| + 1$ and $|a_n^2 + a_n a + a^2| \leq 3a^2 + 3|a| + 1$. Now, let $\epsilon > 0$. Then, since $a_n \rightarrow a$, there exists $N_2 \in \mathbb{N}$ such that, for all $n \geq N_2$,

$$|a - a_n| < \frac{\epsilon}{3a^2 + 3|a| + 1}.$$

Let $N = \max(N_1, N_2)$

$$\begin{aligned} |a_n^3 - a^3| &= |(a_n - a)(a_n^2 + a_n a + a^2)| = |a_n - a| \cdot |a_n^2 + a_n a + a^2| \\ &\leq |a_n - a| \cdot (3a^2 + 3|a| + 1) < \frac{\epsilon}{3a^2 + 3|a| + 1} (3a^2 + 3|a| + 1) \leq \epsilon. \end{aligned}$$

Therefore $a_n \rightarrow a$. (3 marks)

- (b) One can take $a_n = (-1)^n$. To show a_n diverges, we can prove it is not Cauchy, to see this pick $\epsilon = 1$. Then, for any $N \in \mathbb{N}$, $|a_{N+1} - a_N| = 2 > \epsilon$.

We claim that $(b_n)_{n=1}^\infty$ converges to 0. Writing $c_n = \sum_{i=1}^n i a_i$, so $b_n = \frac{1}{n^2} c_n$, note that $c_{2n} = -1 + 2 - 3 + 4 - \dots - (2n-1) + 2n = n$ and similarly, $c_{2n+1} = c_{2n} - (2n+1) = -n-1$. It follows that, for all $n \in \mathbb{N}$, $|c_n| \leq n/2 + 1$ and $|b_n| \leq \frac{1}{2n} + \frac{1}{n^2}$ and since the right hand side of this last equality goes to 0, we have $b_n \rightarrow 0$. (6 marks)

- (c) Note that since 1 is a lower bound for B , we have that $\inf(B) \geq 1$ and since $\sup(A)$ is an upper bound for $A \subset (0, \infty)$ which is non-empty, $\sup(A) > 0$. Now, given $a \in A$ and $b \in B$ we have $\sup(A) \geq a$ and $\inf(B) \leq b$ which then implies $\sup(A)/\inf(B) \geq \frac{a}{b}$ so $\sup(A)/\inf(B)$ is an upper bound for $\{\frac{a}{b} : a \in A, b \in B\}$.

Now, suppose c is an upper bound for $\{\frac{a}{b} : a \in A, b \in B\}$, we want to show $\sup(A)/\inf(B) \leq c$. Note that we can find $a_n \in A$ with $a_n \rightarrow \sup(A)$ and $b_n \in B$ with $b_n \rightarrow \inf(B)$, and by the algebra of limits $\frac{a_n}{b_n} \rightarrow \sup(A)/\inf(B)$, however one also has, for all n , $\frac{a_n}{b_n} \leq c$, therefore $\sup(A)/\inf(B) \leq c$. (5 marks)

- (d) Note that, since $a_n \geq 0$, $s_n = \sum_{j=1}^n a_j$ is a monotone sequence and so it suffices to show it is bounded. Let M be the limit of the sequence $(b_i)_{i=1}^\infty$, note that $b_i \uparrow M$, and so for every $n \in \mathbb{N}$ we have $s_n \leq b_n \leq M$ so the s_n are bounded as desired. (3 marks)

(Total: 20 marks)

3. (a) We first claim that

$$a_{n+1} - a_n = \frac{1}{\sqrt{2n(2n+1)}} + \frac{1}{\sqrt{(2n+1)(2n+2)}} - \frac{1}{\sqrt{n(n+1)}} \leq 0,$$

so that the sequence $(a_n)_{n=1}^{\infty}$ is monotonically decreasing.

To see this, note that we have, by using a common denominator,

$$\frac{1}{\sqrt{2n(2n+1)}} + \frac{1}{\sqrt{(2n+1)(2n+2)}} = \frac{\sqrt{2n(2n+1)} + \sqrt{(2n+1)(2n+2)}}{(2n+1)2} \frac{1}{\sqrt{n(n+1)}}$$

Then we use the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ to see that

$$\frac{\sqrt{2n(2n+1)} + \sqrt{(2n+1)(2n+2)}}{(2n+1)2} \leq \frac{2n + (2n+1) + (2n+1) + (2n+2)}{4(2n+1)} = 1.$$

We also have $a_n \geq 0$ so it is bounded below, therefore it is convergent. (6 marks)

(b) We construct a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that, for each $k \in \mathbb{N}$, $|a_{n_k}| \geq \frac{1}{k}$. To do this, we first set $n_1 = \inf\{n \in \mathbb{N} : |a_n| \geq 1\}$.

Note that the set on the right is bounded below and also non-empty since if it was empty, 1 would be an upper bound for $\{|a_n| : n \in \mathbb{N}\}$.

Now, for $k \in \mathbb{N}$ we inductively define $n_{k+1} = \inf\{n \in \mathbb{N}, n > n_k : |a_n| \geq \frac{1}{k+1}\}$. Again, the set on the right is certainly bounded below and it is also non-empty because if it was empty, $\max(\max\{|a_n| : 1 \leq n \leq n_k\}, \frac{1}{k+1})$ would be an upper bound for $\{|a_n| : n \in \mathbb{N}\}$. This gives us the desired subsequence, and so we have $|a_{n_k}| \leq \frac{1}{k}$ and so $a_{n_k} \rightarrow 0$. (5 marks)

(c) Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be bounded sequences.

(i) Note that $\sup_{n \in \mathbb{N}} a_n$ and $\sup_{n \in \mathbb{N}} b_n$ are upper bounds for $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ respectively, therefore for all $m \in \mathbb{N}$,

$$\left(\sup_{n \in \mathbb{N}} a_n\right) + \left(\sup_{n \in \mathbb{N}} b_n\right) \geq a_m + b_m.$$

Since $\sup_{n \in \mathbb{N}} (a_n + b_n)$ is the least upper bound of the $(a_m + b_m : m \in \mathbb{N})$, the desired inequality follows. (4 marks)

(ii) Take $a_n = 1$ for n odd and $a_n = 0$ for n even and $b_n = 1 - a_n$. Then $\left(\sup_{n \in \mathbb{N}} a_n\right) + \left(\sup_{n \in \mathbb{N}} b_n\right) = 2$ while $\sup_{n \in \mathbb{N}} (a_n + b_n) = 1$. (2 marks)

(d) The fact that $\sum_{n=1}^{\infty} a_n$ is convergent is actually irrelevant. Let $s_n = \sum_{i=1}^n b_i$; then $s_n \geq \sum_{i=1}^n \frac{a_1}{i} = a_1 \sum_{i=1}^n \frac{1}{i}$ and since we know that $\sum_{i=1}^n \frac{1}{i}$ diverges to infinity as $n \rightarrow \infty$ (since it is the sequence of partial sums of the harmonic series), the same is true of s_n , so $\sum_{i=1}^{\infty} b_i$ diverges. (3 marks)

(Total: 20 marks)

4. (a) (i) We observe that $|\sin(1/x^3)| \leq 1$ for $x \neq 0$, so for all $x \neq 0$ we have

$$0 \leq |f(x)| = |x^k| \cdot |\sin(1/x^3)| \leq |x|^k$$

and then the squeeze theorem guarantees that $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$. By sequential continuity this proves that f is continuous at 0. (3 marks)

- (ii) We know that $f(x)$ is differentiable at 0 if and only if

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^k \sin(1/x^3)}{x} = \lim_{x \rightarrow 0} x^{k-1} \sin(1/x^3)$$

exists. Again we have $0 \leq |x^{k-1} \sin(1/x^3)| \leq |x^{k-1}|$, and the right side goes to zero as x does provided that $k > 1$, so by the squeeze theorem the limit exists and is zero for all $k \geq 2$. But this limit defines $f'(0)$, so $f'(0) = 0$ whenever $k \geq 2$. (3 marks)

- (b) (i) Let $f(x) = 2e^x - 3 - x^{2k}$. Then $f(0) = -1$ and $f(1) = 2e - 4$, which is strictly positive because

$$2e - 4 = 2(e - 2) = 2 \sum_{k=2}^{\infty} \frac{1}{k!} > 0.$$

Since $f(x)$ is continuous, the intermediate value theorem says that there is some $x \in (0, 1)$ such that $f(x) = 0$, and then $x^{2k} = 2e^x - 3$. (3 marks)

- (ii) Let $p(x) = -x^2 - 4$. Then for all x we have

$$p(x) \leq -4 \text{ and } 2e^x - 3 > -3 > -4,$$

so $p(x) < 2e^x - 3$ and in particular they cannot be equal. (2 marks)

- (c) (i) For each of the following subsets of $\mathbb{R}$:

(1) (A) $(-1, 1)$ is open because it's an open interval. (1 mark)

(2) (B) $\{0\} \cup [2, \infty)$ is a union of two closed sets, hence closed. (1 mark)

(3) (D) $\mathbb{Q}$ is neither open (it doesn't contain a neighborhood $(-\delta, \delta)$ of $0 \in \mathbb{Q}$ because the irrational numbers are dense) nor closed (it doesn't contain the limit of some sequence of rationals x_n such that $x_n \rightarrow \sqrt{2}$). (1 mark)

(4) (B) This set is the integers $\mathbb{Z}$, which is closed. Or: if $x_n \rightarrow x$ satisfies $\sin(\pi x_n) = 0$ for all n , then by sequential continuity $0 = \sin(\pi x_n) \rightarrow \sin(\pi x)$, so $\sin(\pi x) = 0$ as well and hence this set is closed. (1 mark)

(5) (D) This set is $[0, 1)$, which is neither open (it contains no neighborhood of 0) nor closed (it contains $1 - \frac{1}{n}$ for all n , but not $\lim_n(1 - \frac{1}{n}) = 1$). (1 mark)

- (ii) Fix a point $y \in f(U)$, and write $y = f(x)$ for some $x \in U$. Since U is open, there is some $\delta > 0$ such that $(x - \delta, x + \delta) \in U$. Then $f'(t) > 0$ for all $t \in (x - \delta, x + \delta)$, so f is strictly monotone increasing on this interval. In particular the interval

$$V_y := \left(f(x - \frac{\delta}{2}), f(x + \frac{\delta}{2})\right)$$

is open and contains $y = f(x)$, and it is a subset of $f(U)$. Since $f(U)$ is a union of open sets V_y for $y \in f(U)$, it is open as well. (4 marks)

(Total: 20 marks)

5. (a) (i) There are lots of examples, like $\tan^{-1}(x)$ or $\int_0^x e^{-t^2} dt$; the key is to find something whose derivative is never zero. Note that constant functions do *not* work, because every $x \in \mathbb{R}$ is both a local minimum and a local maximum of $f(x) = c$. (2 marks)
- (ii) Rolle's theorem gives some $y_i \in (x_i, x_{i+1})$ such that $f'(y_i) = 0$, for each $i = 0, 1, \dots, n-1$, and then $y_0 < y_1 < \dots < y_{n-1}$. We now claim that $f^{(k)}(x)$ has at least $n+1-k$ distinct zeroes for all $k \geq 1$, having just proved the base case $k = 1$. If it's true for some $k \geq 1$ then we call these zeroes $z_0 < z_1 < \dots < z_{n-k}$; for each $i = 0, 2, \dots, n-k-1$ we have $f^{(k)}(z_i) = f^{(k)}(z_{i+1}) = 0$, so Rolle's theorem provides $w_i \in (z_i, z_{i+1})$ such that $f^{(k+1)}(w_i) = 0$ for $i = 0, 1, \dots, n-k-1$ and we induct. When $k = n$ we see that $f^{(n)}(x)$ has at least one zero, as desired. (4 marks)
- (b) (i) Certainly $f(0) = c \cdot 0 + f(0)$. For $x \neq 0$, the mean value theorem says that $\frac{f(x)-f(0)}{x-0} = f'(t)$ for some $t \in (0, x)$, so $f(x) - f(0) = f'(t)(x-0) = cx$. (2 marks)
- (ii) If $f(x) = 3x$ and $f(y) = 3y$ where $x < y$, then by the mean value theorem there is $t \in (x, y)$ such that $f'(t) = \frac{f(y)-f(x)}{y-x} = \frac{3y-3x}{y-x} = 3$. (3 marks)
- (iii) Since $\tan^{-1}(x)$ has derivative $\frac{1}{1+x^2}$, we know for some $t \in (0, x)$ that

$$\frac{\tan^{-1}(x) - \tan^{-1}(0)}{x - 0} = \frac{1}{1+t^2} \implies \tan^{-1}(x) = \frac{x}{1+t^2}.$$

Since $0 < t < x$ we have $\frac{x}{1+x^2} < \frac{x}{1+t^2} < \frac{x}{1}$, so $\frac{x}{1+x^2} < \tan^{-1}(x) < x$. (3 marks)

- (c) (i) We can write f on $(-1, 1)$ as a convergent power series

$$f(x) = \sum_{n=0}^{\infty} x^{3n} = 1 + x^3 + x^6 + x^9 + x^{12} + \dots$$

Inside the radius of convergence we can repeatedly differentiate f term by term, so

$$f^{(12)}(x) = (12 \cdot 11 \cdot \dots \cdot 1)x^0 + (15 \cdot 14 \cdot \dots \cdot 4)x^3 + \dots$$

Taking $x = 0$, we conclude that $f^{(12)}(0) = 12!$, or $\frac{1}{12!} \cdot f^{(12)}(0) = 1$. (2 marks)

- (ii) We claim that $f(x)$ is convex on $(0, 1)$, because

$$f'(x) = \frac{3x^2}{(1-x^3)^2} \implies f''(x) = \frac{6x(1-x^3)^2 - 3x^2 \cdot (-6x^2)(1-x^3)}{(1-x^3)^4}$$

which simplifies to

$$f''(x) = \frac{6x(1-x^3) + 18x^4}{(1-x^3)^3} = \frac{6x(1+2x^3)}{(1-x^3)^2},$$

and on $[0, 1)$ the numerator is nonnegative and the denominator is strictly positive, so $f''(x) \geq 0$. Now by convexity we have for $x \in (0, 1)$ the inequality

$$\frac{f(0) + f(x)}{2} \geq f\left(\frac{0+x}{2}\right) \implies \frac{f(x) + 1}{2} \geq f\left(\frac{x}{2}\right),$$

which is what we wanted to show. (4 marks)

(Total: 20 marks)

6. (a) (i) Since f is increasing on $[0, 1]$, we have

$$\begin{aligned} L(f, P_n) &= \sum_{i=0}^{n-1} \left(\inf_{x \in [\frac{i}{n}, \frac{i+1}{n}]} x^r \right) \left(\frac{i+1}{n} - \frac{i}{n} \right) = \sum_{i=0}^{n-1} \left(\frac{i}{n} \right)^r \left(\frac{1}{n} \right) \\ &= \frac{1}{n^{r+1}} \sum_{i=0}^{n-1} i^r = \frac{1}{n^{r+1}} (0^r + 1^r + 2^r + \cdots + (n-1)^r). \end{aligned}$$

Similarly f is maximized on $[\frac{i}{n}, \frac{i+1}{n}]$ at $x = \frac{i+1}{n}$, and so

$$\begin{aligned} U(f, P_n) &= \sum_{i=0}^{n-1} \left(\sup_{x \in [\frac{i}{n}, \frac{i+1}{n}]} x^r \right) \left(\frac{i+1}{n} - \frac{i}{n} \right) = \sum_{i=0}^{n-1} \left(\frac{i+1}{n} \right)^r \left(\frac{1}{n} \right) \\ &= \frac{1}{n^{r+1}} \sum_{i=0}^{n-1} (i+1)^r = \frac{1}{n^{r+1}} (1^r + 2^r + \cdots + n^r). \end{aligned}$$

(2 marks)

- (ii) Comparing term by term gives $U(f, P_n) - L(f, P_n) = \frac{1}{n^{r+1}}(n^r) = \frac{1}{n}$. (2 marks)

- (iii) We know that $f(x)$ is integrable on $[0, 1]$, with integral $\frac{1}{r+1}$, and we have an inequality

$$L(f, P_n) \leq \int_0^1 f(x) dx \leq U(f, P_n), \text{ from which}$$

$$0 \leq U(f, P_n) - \int_0^1 f(x) dx \leq U(f, P_n) - L(f, P_n) = \frac{1}{n}.$$

Evaluating the expression $U(f, P_n) - \int_0^1 f(x) dx$, these inequalities become

$$0 \leq \frac{1^r + 2^r + \cdots + n^r}{n^{r+1}} - \frac{1}{r+1} \leq \frac{1}{n}$$

and we multiply through by n^{r+1} to get the desired inequality. (2 marks)

- (b) (i) Since $x^3 \geq x$ for all $x \geq 1$, we have $0 < e^{-x^3} \leq e^{-x}$ for all $x \geq 1$. The improper integral $\int_1^\infty e^{-x} dx = \frac{1}{e}$ exists, hence by comparison so does $\int_1^\infty e^{-x^3} dx$. (3 marks)

- (ii) Fix $\epsilon > 0$. There is some $N > 0$ such that $x \geq N$ implies $1 - \epsilon < \frac{f(x)}{x} < 1 + \epsilon$, or equivalently $(1 - \epsilon)x < f(x) < (1 + \epsilon)x$, and integrating from N to x gives

$$(1 - \epsilon) \int_N^x t dt \leq \int_N^x f(t) dt \leq (1 + \epsilon) \int_N^x t dt.$$

We evaluate $\int_N^x t dt = \frac{1}{2}(x^2 - N^2)$ and add $C = \int_0^N f(t) dt$ to each side to get

$$(1 - \epsilon) \left(\frac{x^2 - N^2}{2} \right) + C \leq \int_0^x f(t) dt \leq (1 + \epsilon) \left(\frac{x^2 - N^2}{2} \right) + C.$$

Setting $C' = C - (1 - \epsilon)\frac{N^2}{2}$ and $C'' = C - (1 + \epsilon)\frac{N^2}{2}$, and dividing by x^2 , we have

$$\frac{1 - \epsilon}{2} + \frac{C'}{x^2} \leq \frac{1}{x^2} \int_0^x f(t) dt \leq \frac{1 + \epsilon}{2} + \frac{C''}{x^2}$$

for all $x \geq N$. As $x \rightarrow \infty$ the outer bounds converge to $\frac{1-\epsilon}{2}$ and $\frac{1+\epsilon}{2}$, so

$$\exists N' > 0 \text{ such that } x > N' \implies \frac{1}{2} - \epsilon < \frac{1}{x^2} \int_0^x f(t) dt < \frac{1}{2} + \epsilon.$$

Since we can find such an N' for any ϵ , we have $\frac{1}{x^2} \int_0^x f(t) dt \rightarrow \frac{1}{2}$. (3 marks)

- (c) (i) The fundamental theorem of calculus says that $f'(x) = \sin(\sqrt{x^2 + 1})$ for all $x > 0$, and then we use the chain rule to compute

$$f''(x) = \cos(\sqrt{x^2 + 1}) \cdot \frac{x}{\sqrt{x^2 + 1}}.$$

At a local minimum or maximum we have $\sin(\sqrt{x^2 + 1}) = f'(x) = 0$, so $\sqrt{x^2 + 1} = k\pi$ for some positive $k \in \mathbb{Z}$, or $x = \sqrt{k^2\pi^2 - 1}$. Then the second derivative is

$$f''(\sqrt{k^2\pi^2 - 1}) = \cos(k\pi) \cdot \frac{\sqrt{k^2\pi^2 - 1}}{k\pi},$$

which is positive if k is even and negative if k is odd, so by the second derivative test:

$$x = \sqrt{k^2\pi^2 - 1} \text{ is a local } \begin{cases} \text{minimum,} & k \geq 1 \text{ odd} \\ \text{maximum,} & k \geq 2 \text{ even.} \end{cases}$$

(4 marks)

- (ii) If the limit existed and were equal to L , then for every $\epsilon > 0$ there would be an $N > 0$ such that $x > N$ implies $|f(x) - L| < \frac{\epsilon}{2}$, so given $y > x > N$ we would have

$$|f(y) - f(x)| \leq |f(y) - L| + |L - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The mean value theorem says that there's some $t \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(t) = (y - x)\sin(\sqrt{t^2 + 1}),$$

hence $(y - x)|\sin(\sqrt{t^2 + 1})| < \epsilon$. We choose convenient values of x and y : let

$$\left(\sqrt{x^2 + 1}, \sqrt{y^2 + 1}\right) = \left(2\pi n + \frac{\pi}{6}, 2\pi n + \frac{5\pi}{6}\right)$$

for k large enough to ensure $x > N$, and then we have

$$y - x > \sqrt{y^2 + 1} - \sqrt{x^2 + 1} = \frac{2\pi}{3}$$

while $\sin(\sqrt{t^2 + 1}) > \frac{1}{2}$ for all $t \in (x, y)$. We conclude that

$$\epsilon > (y - x)|\sin(\sqrt{t^2 + 1})| > \frac{2\pi}{3} \cdot \frac{1}{2} = \frac{\pi}{3},$$

which is a contradiction, so $\lim_{x \rightarrow \infty} f(x)$ cannot exist after all.

(4 marks)

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add

ExamModuleCode	QuestionNumber	Comments for Students
MATH 40002	1	Part a,b,c,d) The large majority of students did well on all of these parts - thinking in terms of examples/counter-examples is a good approach here, and this will be an important skill as you continue to learn Analysis. Part e) Performance on this part was lower. Many students used the ratio test or the root test, but the problem with these tests is that the limit you want to look at might not exist! It is better to use the definition of the radius of convergence (we are inside the radius of convergence whenever the terms go to 0), or the fact that if you take a "Hadamard product" (this just means term by term li, as opposed to the cauchy product) of absolutely convergent series, then the new series is also absolutely convergent - this is not hard to prove using a comparison test.
MATH 40002	2	On the whole students did well on this question, however many many scripts contained a lot of sloppiness. To give you an example, the question about $a_n \rightarrow a$ implies $a_n^3 \rightarrow a^3$. Imagine all of the a_n and a are equal zero (assuming you didn't do the $a=0$ case separately). Take a look at your solution. How many times did you divide by zero? How many times did you write $X < Y$ and in this case it turns out to a claim that $0 < 0$? Also a lot of people wrote inequalities which were false if $a < 0$. I tended to take marks off for this sloppiness. A lot of people couldn't figure out what to prove in (d), it just seemed "obvious" to some people. Take a look at the model solution!

MATH 40002	3	Many students had problems to fully answer question a). Many solutions were correct in the sense that the students tried to proof convergence by showing that the sequence a_n is monoton and bounded, but in many cases the proof on monotonicity was simply too vague. Moreover, many students argued that a_n is increasing and bounded although it is in fact the other way round. As to question b), many students got the idea right in how to construct a monotonically increasing subsequence, but in many cases this construction was too vague to achieve full marks. Concerning, ci) some students confused boundedness with limit, but many got the idea right in how to proof the inequality. Finally, for d) most of the students used appropriately the fact that $1/i$ is a divergent sequence, and a significant number of students correctly applied the ratio test to show that b_n diverges.
MATH 40002	4	Questions ai) and aii) were, in general, very well answered, mostly very similar to the example solution. As to bii), many students selected a linear polynomial, which made the proof much more complicated than necessary when selecting, for instance, $p = -x^2 - 4$. Nevertheless, these proofs were done most of the time correctly. Ci) was done well too, in general. Finally, most of those students who answered cii) understood rather well the concept of openness and of differential and continuous functions and answered the question very well.
MATH 40002	5	In Q 5ai one has to keep in mind the conditions of boundedness and continuity. Q5aii is done by induction using Rolle or mean value thm, but the process should be explained as clearly as possible. Q5b are easy applications of mean value thm. In Q 5ci one has to use the Taylor exansion of the function. There are several ways to show 5cii either by convexity or more direct comparison.

MATH 40002		(a)(iii) Deducing this from the Darboux sums is significantly easier than trying to show this inequality directly. (b)(i) I need to see some evidence that you knew that the integrand was positive, or that the integral was monotone increasing, to award full marks for using the comparison test. If you wanted to show that the integral existed by examining upper and lower sums of an increasingly large finite interval, you needed to show that the difference between the sums went to zero, not to a positive constant. If you showed that it went to a non-zero constant, you did not rule out the possibility that the integral oscillated, like the function f from (c). (b)(ii) The most elegant method for solving this was to use l'Hôpital's rule, but you needed to explicitly check that the conditions for using it held. It's possible to prove this by directly using the fact about the limit of $f(x)/x$, but that requires turning that fact into an ϵ statement very early to be sufficiently rigorous. (c)(ii) This was a very tough question, well done to anyone who got any marks for it!
------------	--	---