

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)**  
**May-June 2022**

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

**Analysis 1**

Date: 19 May 2022

Time: 09:00 – 12:00 (BST)

Time Allowed: 3:00 hours

Upload Time Allowed: 45 minutes

**This paper has 6 Questions.**

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

**SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.**

1. (a) State whether the following statements are True or False - if it is true give a proof, and if it is false give a counterexample (with justification of why your counterexample works).

(i) Let  $(a_n)_{n=1}^{\infty}$  be an unbounded sequence of non-zero real numbers. **True/False:**  $\frac{1}{a_n} \rightarrow 0$ . (3 marks)

(ii) Let  $A \subset \mathbb{R}$  be non-empty and bounded, with the property that for every  $a, b \in A$  with  $a \neq b$ , one has  $|a - b| \geq \frac{1}{2}$ . **True/False:** We must have  $\sup(A) \in A$ . (3 marks)

(iii) Let  $(a_n)_{n=1}^{\infty}$  be a real sequence with  $\sum_{n=1}^{\infty} |a_n|$  convergent. Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be an injection.

**True/False:**  $\sum_{n=1}^{\infty} a_{f(n)}$  must be convergent. (3 marks)

(b) For each of the sets below, state whether they are either countably infinite or uncountably infinite, and give a rigorous argument to justify your assertion.

(i) The set of all functions  $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  periodic in both variables, that is those  $f$  for which there exists  $m, n \in \mathbb{N}$  such that, for  $i, j \in \mathbb{Z}$ ,  $f(i + m, j + n) = f(i, j)$ . (3 marks)

(ii) The set of all functions  $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  periodic in the first variable, that is those  $f$  for which there exists  $m \in \mathbb{N}$  such that, for  $i, j \in \mathbb{Z}$ ,  $f(i + m, j) = f(i, j)$ . (3 marks)

(c) (i) Let  $A = \left\{ (-1)^n \left( 1 - \frac{1}{n} \right) : n \in \mathbb{N} \right\}$ , find, with proof,  $\sup(A)$ . (3 marks)

(ii) Suppose that  $\sum_{n=1}^{\infty} a_n$  is convergent. Prove that  $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$  is convergent. (2 marks)

(Total: 20 marks)

2. (a) For each part below, give an example of a sequence  $(a_n)_{n=1}^{\infty}$  with the following properties (no proofs necessary).

(i) Bounded, but not Cauchy. (2 marks)

(ii) Not monotone, and has no convergent subsequences. (2 marks)

(iii) All  $a_n$  irrational, and, for every  $a \in \mathbb{R}$ , has a subsequence converging to  $a$ . (2 marks)

(b) Let  $(a_n)_{n=1}^{\infty}$  be a positive sequence with  $\sum_{n=1}^{\infty} a_n$  convergent. Define  $(r_n)_{n=1}^{\infty}$  via  $r_n = \sum_{j=n}^{\infty} a_j$ .

(i) **Carefully** prove  $(r_n)_{n=1}^{\infty}$  is monotone decreasing and converges to 0. (2 marks)

(ii) Prove that  $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}}$  is convergent. (3 marks)

(iii) Prove that  $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$  is divergent. (6 marks)

(iv) Suppose that we also know that  $\sum_{n=1}^{\infty} r_n < \infty$ . Prove that  $na_n \rightarrow 0$ . (3 marks)

(Total: 20 marks)

3. (a) (i) Suppose  $(a_n)_{n=1}^{\infty}$  satisfies  $a_1 = 0$  and, for all  $n \in \mathbb{N}$ ,  $a_{n+1} = \sqrt{6 + a_n}$ . Prove  $(a_n)_{n=1}^{\infty}$  converges and identify its limit. (3 marks)
- (ii) Suppose  $(a_n)_{n=1}^{\infty}$  satisfies  $a_1 = \frac{1}{2}$ ,  $a_2 = 1$ , and, for all  $n \in \mathbb{N}$ ,  $|a_{n+2} - a_{n+1}| \leq |a_{n+1} - a_n|^2$ . Prove that  $(a_n)_{n=1}^{\infty}$  converges. (4 marks)
- (b) For the following two problems on the convergence of sequences, write a proof based on first principles, that is directly use the definition of convergence for sequences.
- (i) Prove that  $\frac{2n}{2n + (-1)^n} \rightarrow 1$ . (3 marks)
- (ii) Suppose that  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . Prove that  $\max(a_n, b_n) \rightarrow \max(a, b)$ . (4 marks)
- (c) Let  $(a_n)_{n=1}^{\infty}$  be a non-negative sequence such that, for every  $n \in \mathbb{N}$ ,

$$a_{n+1} \leq a_n + 1/n^2.$$

Prove that the sequence  $(a_n)_{n=1}^{\infty}$  is convergent. (6 marks)

(Total: 20 marks)

4. (a) Consider the function  $f(x) = \cos(\frac{1}{x})$ , defined on  $\mathbb{R} \setminus \{0\}$ .
- (i) Prove that  $f$  is continuous at all  $x \neq 0$ . (2 marks)
- (ii) Is there a real number  $c$  such that the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$g(x) = \begin{cases} f(x), & x \neq 0 \\ c, & x = 0 \end{cases}$$

is continuous at  $x = 0$ ? Prove your answer. (3 marks)

- (b) (i) Prove that there is some  $x \in \mathbb{R}$  such that  $x = (\cos(x))^3$ . (2 marks)
- (ii) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that  $f(n) = (-1)^n$  for all  $n \in \mathbb{Z}$ . Prove that  $f$  has infinitely many zeroes. (3 marks)
- (c) In each of the following parts we will let  $f$  denote a strictly monotone increasing function.
- (i) Give an example (no proof necessary) of  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a compact set  $K \subset \mathbb{R}$  such that  $f(K)$  is not compact. (3 marks)
- (ii) Prove that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $U \subset \mathbb{R}$  is open, then  $f(U)$  is open. (3 marks)
- (iii) Prove that if  $f : [1, \infty) \rightarrow \mathbb{R}$  is uniformly continuous and  $f(1) = 0$ , then  $\frac{f(x)}{x}$  is bounded. (4 marks)

(Total: 20 marks)

5. (a) For each part below, give an example of a differentiable function  $f : (a, \infty) \rightarrow \mathbb{R}$  with the specified property. Here  $a$  can be anything you like, and you do not need to prove that your examples work.
- (i)  $f'(0) = 0$ , but  $f$  has neither a local minimum nor a local maximum at  $x = 0$ . (2 marks)
  - (ii)  $\lim_{x \rightarrow \infty} f'(x) = 0$  but  $\lim_{x \rightarrow \infty} f(x) = \infty$ . (2 marks)
  - (iii)  $f'(x)$  is unbounded but  $f(x)$  is bounded. (2 marks)
- (b) Define  $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R}$  by  $f(x) = \sqrt{\cos(x)}$ .
- (i) Compute the second-order Taylor polynomial of  $f$ , centered at  $x = 0$ . (3 marks)
  - (ii) Prove the inequality  $\frac{1}{2}(f(0) + f(x)) \leq f(\frac{x}{2})$  for all  $x \in (0, \frac{\pi}{2})$ . (2 marks)
  - (iii) Using a first-order Taylor polynomial, show that  $\sqrt{\cos(\frac{1}{3})} \geq \frac{23}{24}$ . (3 marks)
- (c) (i) Prove that if  $a > b \geq 1$  then  $\frac{1}{b} < \frac{\log(a) - \log(b)}{a - b} < \frac{1}{a}$ . (2 marks)
- (ii) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function whose third derivative  $f'''(x)$  exists on  $(a, b)$ . Suppose that the graph of  $f$  intersects the line from  $(a, f(a))$  to  $(b, f(b))$  at two points  $(p, f(p))$  and  $(q, f(q))$ , where  $a < p < q < b$ . Prove that there is some  $c \in (a, b)$  such that  $f'''(c) = 0$ . (4 marks)

(Total: 20 marks)

6. (a) Let  $f(x) = \log x$ , and define a partition of  $[1, e]$  by  $P_n = (1, e^{1/n}, e^{2/n}, \dots, e^{(n-1)/n}, e)$ .
- (i) Write expressions for the Darboux sums  $L(f, P_n)$  and  $U(f, P_n)$ . You may leave your answers as sums (i.e., you can use  $\sum$  notation), but every summand should be evaluated explicitly. (4 marks)
  - (ii) Find a closed form expression (no  $\sum$ ) for  $U(f, P_n) - L(f, P_n)$ . (2 marks)
  - (iii) Determine (with proof) the value of  $\lim_{n \rightarrow \infty} U(f, P_n)$ . (4 marks)
- (b) (i) Determine whether or not the improper integral  $\int_1^\infty (e^{1/x} - 1) dx$  exists. (3 marks)
- (ii) Prove that  $\int_1^\infty (e^{1/x} - 1)^r dx$  exists for all real  $r > 1$ . (3 marks)
- (c) Suppose that  $f(x) = \sum_{n=0}^\infty a_n x^n$  has radius of convergence  $R > 1$ , and that  $\int_{-1}^1 f(x) x^n dx = 0$  for all integers  $n \geq 0$ . Prove that  $f(x) = 0$  on  $[-1, 1]$ . (4 marks)

(Total: 20 marks)

BSc, MSc and MSci EXAMINATIONS (MATHEMATICS)

May – June 2022

This paper is also taken for the relevant examination for the Associateship of the  
Royal College of Science.

Analysis 1

Date: Thursday, 19th May 2022

Time: 09:00 – 12:00

Time Allowed: 3 Hours

This paper has *6 Questions*.

Statistical tables will not be provided.

- Credit will be given for all questions attempted.
- Each question carries equal weight.

1. (a) (i) **False:** Take  $a_n = n$  for  $n$  even and  $a_n = 1$  for  $n$  odd. Then  $a_n$  is unbounded but  $\frac{1}{a_n} = 1$  for every  $n$  odd so one does not have  $\frac{1}{a_n} \rightarrow 0$ . (3 marks)
- (ii) **True:**  $A$  is bounded and non-empty, so by the completeness axiom  $\sup(A)$  exists. Now, for each  $n \in \mathbb{N}$ , there exists  $a_n \in A$  such that  $a_n + 2^{-n} > \sup(A)$ . Since  $\sup(A)$  is an upper bound for  $A$ , we also have  $a_n \leq \sup(A)$ , so that  $|a_n - \sup(A)| < 2^{-n}$ . By the triangle inequality, for every  $n \geq 2$  we have  $|a_2 - a_n| \leq |a_n - \sup(A)| + |a_2 - \sup(A)| < 2^{-n} + 2^{-2} \leq 1/2$ . Therefore we must have, for every  $n \geq 2$ ,  $a_n = a_2$ . However, this means that, for every  $n \geq 1$ ,  $|a_2 - \sup(A)| \leq 2^{-n}$ , which forces  $|a_2 - \sup(A)| = 0$ , so  $\sup(A) = a_2 \in A$ . (3 marks)
- (iii) **True:** Since absolute convergence implies convergence, it suffices to show  $\sum_{n=1}^{\infty} |a_{f(n)}|$  is convergent. Since  $\sum_{n=1}^{\infty} |a_{f(n)}|$  is a series of positive terms, it suffices to show its partial sums are bounded above, we claim they are bounded above by the value of the convergent series  $\sum_{n=1}^{\infty} |a_n|$ . Given any  $N \in \mathbb{N}$ , let  $M = \max\{f(i) : 1 \leq i \leq N\}$ . Since  $f$  is an injection,  $\sum_{n=1}^N |a_{f(n)}| \leq \sum_{n=1}^M |a_n|$ , the key point is that each term in the right sum appears at most once in the left sum. We are done since  $\sum_{n=1}^M |a_n| \leq \sum_{n=1}^{\infty} |a_n|$ . (3 marks)
- (b) (i) **Uncountably infinite:** Writing  $A$  for the set in the problem, and  $G = \{f : \mathbb{Z} \rightarrow \mathbb{Z}\}$ , we claim there is an injection from  $G$  into  $A$  given by  $G \ni g \mapsto f \in A$  with  $f(i, j) = (0, g(i - j))$ . Since  $f(i + 1, j + 1) = f(i, j)$  so it is indeed true that  $f \in A$ . Therefore it suffices to show  $G$  is uncountable, but this follows from Cantor's diagonal argument - given an enumeration of  $G$ , written  $g_1, \dots, g_n, \dots$  we can define a new  $\tilde{g} \in G$  with  $\tilde{g} \neq g_n$  for all  $n \in \mathbb{N}$  by setting  $\tilde{g}(i) = 0$  for  $i \leq 0$  and  $\tilde{g}(i) = g_i(i) + 1$  for  $i \geq 1$ . (3 marks)
- (ii) **Uncountably infinite:** Writing  $B$  for the set in the problem, and defining  $G$  as in the answer above, we note that there is an injection from  $G$  into  $B$  given by  $G \ni g \mapsto f \in B$  with  $f(i, j) = (0, g(j))$ . Note that  $f(i + 1, j) = f(i, j)$  so  $f \in B$  as desired, we are then done since  $G$  is uncountably infinite (argued above). (3 marks)
- (c) (i) We claim  $\sup(A) = 1$ . Clearly 1 is an upper bound since  $|1 - 1/n| < 1$ . To show that 1 is indeed the supremum we show that, for every  $\epsilon > 0$ , there exists  $a \in A$  with  $a + \epsilon > 1$ . Given  $\epsilon > 0$ , we choose  $N = \lceil \epsilon^{-1} \rceil$  so that  $\frac{1}{N} < \epsilon$ . We can then  $(-1)^{2N}(1 - \frac{1}{2N}) \in A$  and  $(-1)^{2N}(1 - \frac{1}{2N}) = 1 - \frac{1}{2N} > 1 - \epsilon$ . (3 marks)
- (ii) Since  $\sum_{n=1}^{\infty} a_n$  is convergent, one has  $a_n \rightarrow 0$  so  $(a_n)_{n=1}^{\infty}$  is bounded and there exists  $M > 0$  such that, for all  $n \in \mathbb{N}$ ,  $|a_n| \leq M$ . Then  $|a_n/n^2| \leq M/n^2$ , so  $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$  absolutely converges by comparison to  $\sum_{n=1}^{\infty} \frac{M}{n^2} = M \sum_{n=1}^{\infty} \frac{1}{n^2}$  (convergent by the  $p$ -test). (2 marks)
- (Total: 20 marks)

2. (a) (i) Any bounded sequence which doesn't converge works, like  $a_n = (-1)^n$ . (2 marks)
- (ii) One can take  $a_n = (-1)^n n$ . (2 marks)
- (iii) Let  $(b_n)_{n=1}^\infty$  be an enumeration of the countable set  $\mathbb{Q}$ , the solutions to the January test verified that every real number can be obtained as a subsequential limit of  $(b_n)_{n=1}^\infty$ . Since every real number  $r$  can be written  $r = \sqrt{3} + u$  for another real  $u$ , we can take  $a_n = \sqrt{3} + b_n$  to get an irrational sequence with the desired property. (2 marks)

- (b) (i) Writing  $s_n = \sum_{j=1}^n a_j$  we have that  $s_n$  is monotone increasing and converges to  $\sum_{j=1}^\infty a_j$ . On the other hand,  $r_n = \lim_{m \rightarrow \infty} \sum_{j=n}^m a_j = \lim_{m \rightarrow \infty} s_m - s_{n-1} = \left( \sum_{j=1}^\infty a_j \right) - s_{n-1}$ , so we see  $r_n$  is monotone decreasing and converges to 0 as  $n \rightarrow \infty$ . (2 marks)

- (ii) Note that, for  $n \in \mathbb{N}$

$$\frac{a_n}{\sqrt{r_n}} = \frac{r_n - r_{n+1}}{\sqrt{r_n}} = \frac{(\sqrt{r_n} - \sqrt{r_{n+1}})(\sqrt{r_n} + \sqrt{r_{n+1}})}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}),$$

where, in the last inequality, we used that  $r_{n+1} < r_n$ . The series  $\sum_{n=1}^\infty 2(\sqrt{r_n} - \sqrt{r_{n+1}})$  is a convergent series since it is telescoping, and so by comparison  $\sum_{n=1}^\infty \frac{a_n}{\sqrt{r_n}}$  also converges. (4 marks)

- (iii) It suffices to show that, for any fixed  $M \geq 1$ , one can find  $N \in \mathbb{N}$  such that  $\sum_{n=M}^N \frac{a_n}{r_n} \geq \frac{1}{2}$ , this would show that the sequence of partial sums for  $\sum_{n=1}^\infty \frac{a_n}{r_n}$  is not Cauchy. Note that

$$\sum_{n=M}^K \frac{a_n}{r_n} \geq \sum_{n=M}^K \frac{a_n}{r_M} = \frac{r_M - r_{K+1}}{r_M} = 1 - \frac{r_{K+1}}{r_M}.$$

For fixed  $M$ ,  $\lim_{K \rightarrow \infty} \frac{r_{K+1}}{r_M} = 0$ , so by taking  $N$  large enough we can guarantee  $\frac{r_{N+1}}{r_M} < \frac{1}{2}$  and  $1 - \frac{r_{N+1}}{r_M} \geq \frac{1}{2}$ . (6 marks)

- (iv) We claim  $\sum_{n=1}^\infty na_n$  is convergent, if we show this then the desired result follows by the divergence test. Since  $na_n \geq 0$ , it suffices to show that there exists  $M > 0$  such that, for all  $N \in \mathbb{N}$ , one has  $\sum_{n=1}^N na_n \leq M$ . To see  $M = \sum_{n=1}^\infty r_n$  works, note that

$$\sum_{n=1}^N na_n = \sum_{j=1}^N \sum_{n=j}^N a_n \leq \sum_{j=1}^N r_j \leq \sum_{j=1}^\infty r_j.$$

(2 marks)

(Total: 20 marks)

3. (a) (i) First, we prove by induction that, for all  $n \in \mathbb{N}$ ,  $0 \leq a_n \leq 3$ . The base case  $n = 1$  is immediate. For the inductive step, supposing  $0 \leq a_n \leq 3$  we have  $0 \leq (6 + a_n) \leq 9$ , so  $0 \leq a_{n+1} \leq 3$  as desired. Second, we prove by induction that for all  $n \in \mathbb{N}$ ,  $a_{n+1} \geq a_n$ . This is true for  $n = 1$  since  $a_2 = \sqrt{6} > 0 = a_1$ . For the inductive step,  $a_{n+1} \geq a_n$  implies  $a_{n+2} = \sqrt{6 + a_{n+1}} \geq \sqrt{6 + a_n} = a_{n+1}$ . It follows that  $a_n$  is bounded above and monotone increasing, so it must be convergent. (3 marks).

(ii) We prove by induction that, for all  $n \in \mathbb{N}$ , one has  $|a_{n+1} - a_n| \leq 2^{-n}$ . This is clear for  $n = 1$ . For the inductive step, suppose  $|a_{n+1} - a_n| \leq 2^{-n}$ ; then  $|a_{n+2} - a_{n+1}| \leq 2^{-2n} \leq 2^{-n-1}$  as desired. Thus  $\sum_{n=1}^{\infty} (a_{n+1} - a_n)$  is absolutely convergent, by comparison to  $\sum_{n=1}^{\infty} 2^{-n} = 1$ . The sequence of partial sums  $\sum_{j=1}^{n-1} (a_{j+1} - a_j) = a_n - a_1$  (for  $n > 1$ ) is therefore convergent, hence so is  $(a_n)$  as  $n \rightarrow \infty$ . (4 marks)

(b) (i) Let  $\epsilon > 0$ , then choose  $N \in \mathbb{N}$  such  $N > \frac{1}{\epsilon}$ . Then, for any  $n \geq N$ , it follows that

$$\left| \frac{2n}{2n + (-1)^n} - 1 \right| = \frac{1}{2n + (-1)^n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

where in the first and third inequalities we use  $2n + (-1)^n \geq n$  and  $N > \frac{1}{\epsilon}$ . (3 marks)

(ii) Without loss of generality, we assume that  $a \geq b$ , so  $\max(a, b) = a$ . Let  $\epsilon > 0$ . Then there exists  $N \in \mathbb{N}$  such that, for all  $n \geq N$ ,  $|a_n - a| < \epsilon$  and  $|b_n - b| < \epsilon$ . Now fix  $n \geq N$ . If  $a_n \geq b_n$ , it follows that  $|\max(a_n, b_n) - \max(a, b)| = |a_n - a| < \epsilon$ . Supposing instead that  $b_n > a_n$  we have  $b_n > a - \epsilon$  since  $a_n > a - \epsilon$ , and also  $b_n < b + \epsilon \leq a + \epsilon$ . Therefore, we have  $|\max(a_n, b_n) - \max(a, b)| = |b_n - a| < \epsilon$ . (4 marks)

(c) Note that, for any  $N \in \mathbb{N}$ ,

$$0 \leq a_N \leq a_1 + \sum_{j=1}^{N-1} \frac{1}{j^2} \leq a_1 + \sum_{j=1}^{\infty} \frac{1}{j^2}$$

and, since the rightmost series is convergent (by the  $p$ -test), it follows that  $(a_n)_{n=1}^{\infty}$  is bounded. By Bolzano–Weierstrass,  $(a_n)$  has a convergent subsequence, with some limit  $L$ . We show  $a_n \rightarrow L$  as  $n \rightarrow \infty$ . Let  $\epsilon > 0$ , and choose  $I \in \mathbb{N}$  such that, for all  $i \geq I$ ,  $|a_{n_i} - L| < \epsilon/2$  and, for all  $n \geq m \geq I$ , one has  $\sum_{j=m}^n \frac{1}{j^2} < \epsilon/2$  (we can do this because the sequence of partial sums for  $\sum_j \frac{1}{j^2}$  is Cauchy). We then take  $N = n_I \geq I$ . Then we have

$$\forall n \geq N : \quad a_n \leq a_{n_I} + \sum_{j=n_I}^{n-1} \frac{1}{j^2} < (L + \epsilon/2) + \epsilon/2 = L + \epsilon.$$

On the other hand, there exists  $j \geq i$  such that  $n_j \geq n$  and we then have

$$a_n \geq a_{n_j} - \sum_{k=n}^{n_j-1} \frac{1}{k^2} \geq (L - \epsilon/2) - \epsilon/2 = L - \epsilon.$$

Putting these two together gives  $|a_n - L| < \epsilon$  as desired. (6 marks)

(Total: 20 marks)

4. (a) (i) Since  $\cos(x)$  is continuous at all  $x \in \mathbb{R}$ , the composition  $\cos(\frac{1}{x})$  is continuous wherever the function  $x \mapsto \frac{1}{x}$  is continuous, namely at all  $x \neq 0$ . (2 marks)
- (ii) No such  $c$  exists. If it did then the sequence  $x_n = \frac{1}{2\pi n}$  satisfies  $f(x_n) = \cos(2\pi n) = 1$  and  $x_n \rightarrow 0$ , so by sequential continuity we would have  $f(0) = \lim_{n \rightarrow \infty} f(x_n) = 1$ . But similarly the sequence  $y_n = \frac{1}{(2n+1)\pi}$  satisfies  $f(y_n) = -1$  and  $y_n \rightarrow 0$ , so we would have  $f(0) = \lim_{n \rightarrow \infty} f(y_n) = -1$ , so  $f(0)$  would have to be both 1 and  $-1$  and this is impossible. (3 marks)

- (b) (i) Let  $f(x) = x - (\cos(x))^3$ , which is continuous. Since  $-1 \leq (\cos(x))^3 \leq 1$  we have

$$f(-2) = -2 - (\cos(-2))^3 \leq -1 \quad \text{and} \quad f(2) = 2 - (\cos(-2))^3 \geq 1,$$

so  $f(-2) < 0 < f(2)$ . Then by the intermediate theorem there is some  $x \in (-2, 2)$  such that  $f(x) = 0$ , or equivalently  $x = (\cos(x))^3$ . (2 marks)

- (ii) On every interval  $[n, n+1]$  we have either  $f(n) = 1$  and  $f(n+1) = -1$ , or  $f(n) = -1$  and  $f(n+1) = 1$ . Either way 0 lies strictly between  $f(n)$  and  $f(n+1)$ , so by the intermediate value theorem there is some  $x \in (n, n+1)$  such that  $f(x) = 0$ . We can find such an  $x$  for every integer  $n$ , so there are infinitely many zeroes. (3 marks)

- (c) (i) Define (for example) the piecewise function  $f(x) = \begin{cases} x, & x \leq 0 \\ x+1, & x > 0. \end{cases}$  Then  $K = [0, 1]$

is compact, but  $f(K) = \{0\} \cup (1, 2]$  is not compact because it is not closed: it contains the convergent sequence  $(1 + \frac{1}{n})_{n \geq 1}$ , but not its limit 1. (3 marks)

- (ii) **Solution 1:** Take  $y \in f(U)$ , and write  $y = f(x)$  where  $x \in U$ . Since  $U$  is open, it contains the open interval  $(x - \delta, x + \delta)$  for some  $\delta > 0$ . But then  $f(U)$  contains both  $a = f(x - \frac{\delta}{2})$  and  $b = f(x + \frac{\delta}{2})$ ; since  $f$  is strictly increasing we have  $a < y < b$ . Now  $f$  is continuous and  $[x - \frac{\delta}{2}, x + \frac{\delta}{2}] \subset U$ , so the intermediate value theorem tells us that  $f(U)$  must contain the entire interval  $[a, b]$ . Thus if we take  $\delta' = \min(b - y, y - a)$  then  $\delta' > 0$  and  $f(U)$  contains the open interval  $(y - \delta', y + \delta')$ . We can find such a  $\delta'$  for any  $y \in f(U)$ , so  $f(U)$  is open.

**Solution 2:** We know that  $f$  is injective, with continuous inverse  $g : f(\mathbb{R}) \rightarrow \mathbb{R}$ , and that  $f(\mathbb{R})$  is an open interval. We have  $f(U) = g^{-1}(U)$ , and since  $g$  is continuous and  $U$  is open the preimage  $g^{-1}(U)$  is open (see #8 on PS1), so  $f(U)$  is open. (3 marks)

- (iii) By definition, given  $\epsilon = 1$  there is  $\delta > 0$  such that for all  $x, y \geq 1$ ,  $|x - y| < \delta$  implies  $|f(x) - f(y)| < 1$ ; we can shrink  $\delta$  if needed so that  $\delta < 1$ . Given  $x \geq 1$ , we write  $x = q\delta + r$  with  $q \in \mathbb{Z}$  and  $0 \leq r < \delta$ ; we note that  $q \geq 1$ , since otherwise  $x < \delta < 1$ . Then  $x < q\delta + 1$  implies  $\frac{x-1}{q} < \delta$ , so by induction

$$f(x) = f(1 + q(\frac{x-1}{q})) < f(1 + (q-1)(\frac{x-1}{q})) + 1 < \cdots < f(1) + q = q.$$

But then  $\frac{f(x)}{x} \leq \frac{q}{q\delta + r} \leq \frac{1}{\delta}$ , so we have  $0 \leq \frac{f(x)}{x} \leq \frac{1}{\delta}$  for all  $x \geq 1 + \delta$ . (4 marks)

(Total: 20 marks)

5. (a) (i)  $f(x) = x^3$ . (2 marks)  
 (ii)  $f(x) = \log(x)$ , defined on  $(0, \infty)$ , since  $f'(x) = \frac{1}{x}$ . (2 marks)  
 (iii)  $f(x) = \sin(x^2)$ , since  $|f(x)| \leq 1$  but  $f'(x) = 2x \cos(x^2)$  satisfies  $f'(\sqrt{2\pi n}) = 2\sqrt{2\pi n} \rightarrow \infty$ . (2 marks)

- (b) (i) Writing  $f(x) = (\cos(x))^{1/2}$ , we compute that  $f'(x) = -\frac{1}{2} \sin(x)(\cos(x))^{-1/2}$  and

$$f''(x) = -\frac{1}{2}(\cos(x))^{1/2} - \frac{1}{4} \sin^2(x)(\cos(x))^{-3/2}.$$

So the second-order Taylor polynomial is

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 + \frac{0}{1}x + \frac{-1/2}{2}x^2 = \boxed{1 - \frac{x^2}{4}}.$$

(3 marks)

- (ii) We write  $f''(x)(\cos(x))^{3/2} = -\frac{1}{2}(\cos(x))^2 - \frac{1}{4}(\sin(x))^2 = -\frac{1}{4} - \frac{1}{4}(\cos(x))^2$ . The right side is negative, so  $f''(x) < 0$ , and thus  $-f$  is convex. By the definition of convexity we get  $\frac{1}{2}(-f(0)) + \frac{1}{2}(-f(x)) \geq (-f(\frac{x}{2}))$ , and we multiply through by  $-1$  to get the desired inequality. (2 marks)

- (iii) Taylor's theorem says that  $f(\frac{1}{3}) = 1 + \frac{f''(c)}{2}(\frac{1}{3})^2$  for some  $c \in (0, 1)$ . Using the computation of  $f''(c)$  from part (a), together with the triangle inequality, we have

$$|f''(c)| \leq \left| \frac{1}{2}(\cos(c))^{1/2} \right| + \left| \frac{1}{4} \sin^2(c) \cos(c)^{-3/2} \right| \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

So the error term has absolute value at most  $\frac{3/4}{2}(\frac{1}{3})^2 = \frac{1}{24}$ , and thus  $\sqrt{\cos(\frac{1}{3})} = f(\frac{1}{3})$  is at least  $1 - \frac{1}{24} = \frac{23}{24}$ . (The actual value is  $\frac{23.330...}{24}$ .) (3 marks)

- (c) (i) We apply the mean value theorem to get  $\frac{\log(a) - \log(b)}{a - b} = \frac{1}{c}$  for some  $c \in (a, b)$ . Then  $a < c < b$  implies that  $\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$ , so the inequality follows immediately. (2 marks)  
 (ii) Let  $m$  be the slope of the line from  $(a, f(a))$  to  $(b, f(b))$ . Then we have

$$m = \frac{f(p) - f(a)}{p - a} = \frac{f(q) - f(p)}{q - p} = \frac{f(b) - f(q)}{b - q}$$

since these points all lie on the same line. Applying the mean value theorem to  $f$  on each interval  $[a, p]$ ,  $[p, q]$ , and  $[q, b]$ , we get

$$x \in (a, p), y \in (p, q), z \in (q, b) \text{ such that } f'(x) = f'(y) = f'(z) = m.$$

Now we apply Rolle's theorem to  $f'$  on each of the intervals  $[x, y]$  and  $[y, z]$  to get

$$s \in (x, y), t \in (y, z) \text{ such that } f''(s) = f''(t) = 0,$$

and then to  $f''$  on  $[s, t]$  to get  $c \in (s, t)$  such that  $f'''(c) = 0$ . (4 marks)

(Total: 20 marks)

6. (a) Let  $f(x) = \log x$ , and define a partition of  $[1, e]$  by  $P_n = (1, e^{1/n}, e^{2/n}, \dots, e^{(n-1)/n}, e)$ .

(i) Since  $f(x)$  is increasing, its infimum and supremum on an interval  $[x_i, x_{i+1}]$  are simply  $f(x_i)$  and  $f(x_{i+1})$  respectively. Thus

$$L(f, P_n) = \sum_{i=0}^{n-1} \underbrace{(\log e^{i/n})}_{=f(x_i)} \underbrace{(e^{(i+1)/n} - e^{i/n})}_{=\Delta x_i} = \sum_{i=0}^{n-1} \frac{i}{n} \cdot e^{i/n} (e^{1/n} - 1)$$

and  $U(f, P_n)$  is the same but with  $\log e^{(i+1)/n} = \frac{i+1}{n}$  in place of  $\frac{i}{n}$ . Thus

$$L(f, P_n) = \frac{e^{1/n} - 1}{n} \sum_{i=0}^{n-1} i e^{i/n}, \quad U(f, P_n) = \frac{e^{1/n} - 1}{n} \sum_{i=0}^{n-1} (i+1) e^{i/n}.$$

(4 marks)

(ii) Using the Darboux sums from (i), the difference is

$$U(f, P_n) - L(f, P_n) = \frac{e^{1/n} - 1}{n} \sum_{i=0}^{n-1} e^{i/n} = \frac{e^{1/n} - 1}{n} \cdot \frac{e - 1}{e^{1/n} - 1} = \boxed{\frac{e - 1}{n}}.$$

(Here we have simplified  $\sum e^{i/n}$  as a partial sum of a geometric series.) (2 marks)

(iii) Since  $U(f, P_n) - L(f, P_n) \rightarrow 0$  as  $n \rightarrow \infty$ , we know that  $\lim_{n \rightarrow \infty} U(f, P_n)$  is equal to

$$\int_1^e \log(x) dx = [x \log(x) - x]_{x=1}^{x=e} = (e - e) - (0 - 1) = 1.$$

Here we can use e.g. integration by parts to evaluate the integral. (4 marks)

(b) (i) We apply the inequality  $e^y \geq 1 + y$  for  $y \geq 0$ , which follows directly from the power series definition of  $e^y$ . Letting  $y = \frac{1}{x}$  gives the inequality  $e^{1/x} - 1 \geq \frac{1}{x}$ , so for any  $b > 1$  we have

$$\int_1^b (e^{1/x} - 1) dx \geq \int_1^b \frac{1}{x} dx = \log(b).$$

This goes to infinity as  $b$  does, so the improper integral does not exist. (3 marks)

(ii) Given  $x \geq 1$ , we apply the mean value theorem to  $f(y) = e^y$  on  $[0, \frac{1}{x}]$  to see that

$$\frac{e^{1/x} - e^0}{1/x - 0} = e^c \text{ for some } c \in (0, \frac{1}{x}).$$

Then  $c < \frac{1}{x} \leq 1$ , so  $e^c < e$  and we rearrange to get  $e^{1/x} - 1 < \frac{e}{x}$ . Raising each side to the  $r$ th power, we have  $0 < (e^{1/x} - 1)^r < \frac{e^r}{x^r}$  for all  $x \geq 1$ , and the improper integral

$$\int_1^\infty \frac{e^r}{x^r} dx = \left[ -\frac{e^r}{(r-1)x^{r-1}} \right]_{x=1}^\infty = \frac{e^r}{r-1}$$

exists since  $r > 1$ , so the improper integral  $\int_1^\infty (e^{1/x} - 1)^r dx$  exists as well. (3 marks)

(c) We note that  $f$  is continuous (hence bounded) on  $[-1, 1]$ . Write  $P_n(x) = \sum_{i=0}^n a_i x^i$ . Then

$$\int_{-1}^1 f(x) P_n(x) dx = \int_{-1}^1 f(x) \left( \sum_{i=0}^n a_i x^i \right) dx = \sum_{i=0}^n a_i \int_{-1}^1 f(x) x^i dx = 0$$

for all  $n \geq 0$ . We claim that  $f(x)P_n(x) \rightarrow f(x)^2$  uniformly on  $[-1, 1]$ ; if this is true then

$$\int_{-1}^1 f(x)^2 dx = \lim_{n \rightarrow \infty} \int_{-1}^1 f(x) P_n(x) dx = \lim_{n \rightarrow \infty} 0 = 0,$$

and  $f(x)^2 \geq 0$  is continuous and nonnegative, so this implies that  $f(x)^2 = 0$  (see PS7#1) and hence  $f(x) = 0$ . Thus we must prove the claim about uniform convergence.

Take  $M \geq 0$  such that  $|f(x)| \leq M$  for all  $x \in [-1, 1]$ . We know that  $P_n \rightarrow f$  uniformly on  $[-1, 1]$  since  $R > 1$ , so given  $\epsilon > 0$ , there is  $N > 0$  such that

$$n \geq N \implies |P_n(x) - f(x)| < \frac{\epsilon}{M} \text{ for all } x \in [-1, 1].$$

But then  $n \geq N$  implies that  $|f(x)P_n(x) - f(x)^2| < |f(x)| \cdot \frac{\epsilon}{M} \leq \epsilon$  for all such  $x$  as well. So  $f(x)P_n(x) \rightarrow f(x)^2$  uniformly on  $[-1, 1]$  as well, completing the proof. (4 marks)

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

| ExamModuleCode       | QuestionNumber | Comments for Students                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
|----------------------|----------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|                      |                | <p>For part a)</p> <p>Most students correctly solved part i), the biggest places students lost points were in part ii) assuming the existence of a maximum of the set A without giving any argument and in part iii) saying was a rearrangement (it may not be because f can fail to be surjective) or only saying that f not being surjective was fine since there are fewer terms in the sum (this is right but shouldn't argued more carefully)</p> <p>For part b)</p> <p>For part)i) the question was worded in a bit of way that was unclear, it was said that f was periodic in both arguments but the relation given was <math>f(i + m, j + n) = f(i, j)</math> for some periods m,n. Note that many students instead interpreted the condition as meaning there exists m and n such that <math>f(i+m, j) = f(i, j)</math> and <math>f(i, j+n) = f(i, j)</math>. However, in the former case the set of f is uncountably infinite and in the latter it is countably infinite. Students who wrote a consistent argument for the latter case were still given full or most points for the problem.</p> <p>For part c), most of the issues were with part ii). You cannot use comparison with <math>\sum_n a_n</math> since</p> <p>1 you do not know <math>\sum_n a_n</math> is absolutely convergent.</p> |
| Analysis 1_MATH40002 |                |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |

---

This question was found extremely hard by the cohort. The majority of scripts scored 5/20, with full marks for a(i) and a(ii), one mark for b(i) (showing  $r_n$  monotone decreasing) and no further marks scored.

Quite a few students wrote a lot of nonsense for b(ii) to b(iv), pages and pages worth 0 marks in a time-critical situation. Furthermore people would write what they were claiming to be proofs. For me, this is bad exam technique. These parts are tough, and the proofs need ideas. It's very easy for a marker to spot whether an argument contains the right idea, and once I was convinced it was not there it was just a case of reading the argument and spotting the error, which was often a basic inequality reversed. If there are students out there who think that writing a page of stuff which starts with the question, continues with something which cheats in one place in the middle, and ends up with a claimed proof at the end -- if there are people who think that this approach is scoring them marks, then they need to learn that a totally wrong technique gets 0 however many pages you write about it. Of course I suspect that many people were not out to deceive and might simply have made a slip and thought that they had done the question. Those using "soft" techniques (basic manipulation of inequalities and little more) and thinking that they'd got one of b(ii) and b(iii) out could have considered the possibility that essentially the same soft argument which "proves" one of these parts also "disproves" the other one!

Many people asserted that a decreasing sequence of real numbers which is bounded below by 0 must tend to 0. This is obviously wrong; take your favourite decreasing sequence of real numbers which tends to 0 and then add 37 to them all. You are lucky that I was not allowed to subtract 2 marks for every time I saw this obviously false claim being used to do b(i).

A common (incorrect) argument was: "if the difference between consecutive elements of a sequence goes to zero then the sequence must converge". This is not true! (Think of a counterexample). So, please review the definition of a Cauchy sequence as this will be of 3 fundamental importance in future courses.

Students who answered 4b(i) also answered 4b(ii), these were the best answered questions.

The formulation of 4c(i) was somewhat difficult to grasp (namely, the question of the domain of  $f$ ), 4 this part was marked more generously.

---

Analysis 1\_MATH40002

---

Analysis 1\_MATH40002

---

Analysis 1\_MATH40002

---

Analysis 1\_MATH40002

---

---

Analysis 1\_MATH40002

---

---

Students mostly did well on parts a, b(i), and c(i). 5a(iii): many students gave one of two unusual answers, both from a popular website for maths questions, but one of these answers was wrong because it had domain  $(-1,1)$ . 5b(ii): most people tried replacing  $f$  with the Taylor polynomial from b(i), which was not a valid step. 5b(iii): this was very hard. Lots of students tried to use a 2nd order Taylor polynomial and ignored the error term, getting a lower bound of  $35/36$  when in fact it was about  $34.995/36$ .

---

Part (a) was generally done well - those who got part (i) tended to get the remainder of the question. Quite a few students did not state that  $\log(x)$  is an increasing function (which was needed to justify values for inf/sup.

Many students did not attempt parts (b) and (c)

Part (b)i) Those who attempted this mostly got that  $e^x - 1 > 1/x$  and were able to conclude the integral diverges.

Part(b)ii) This was done badly by most who attempted, many attempted to integrate by parts rather than find a bound.

Part (c) several students attempted to do this by analysing each term in the sum - many failed to realise you needed to fix the  $n$  in  $f(x)x^n$ ...thus giving wrong arguments.