

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)  
May 2023

This paper is also taken for the relevant examination for the  
Associateship of the Royal College of Science

**Analysis 1**

Date: 18 May 2023

Time: 14:00 – 17:00 (BST)

Time Allowed: 3hrs

**This paper has 6 Questions.**

**Please Answer Each Question in a Separate Answer Booklets**

Candidates should start their answers to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

**DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO**

1. (a) (i) We say a function  $f : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$  is decreasing if  $i \leq j \Rightarrow f(i) \geq f(j)$ . Let  $S$  be the set of functions  $f : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$  which are decreasing. Is  $S$  countably infinite or uncountable? Prove your answer. (5 marks)
- (ii) We say a function  $f : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$  is increasing if  $i \leq j \Rightarrow f(i) \leq f(j)$ . Let  $T$  be the set of functions  $f : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$  which are increasing. Is  $T$  countably infinite or uncountable? Prove your answer. (5 marks)
- (b) For the following problems, you will be asked to prove statements from first principles.
- (i) Prove  $\frac{1}{n^2 + n - 5} \rightarrow 0$  using only the definition of convergence of sequences. (4 marks)
- (ii) Let  $A \subset \mathbb{R}$  be non-empty and bounded. Using only the definition of supremum, prove
- $$y = \sup(A) \iff y \text{ is an upper bound of } A \text{ and } \forall \epsilon > 0, \exists a \in A \text{ such that } a > y - \epsilon.$$
- (6 marks)

(Total: 20 marks)

2. (a) (i) Given a sequence of real numbers  $(a_n)_{n=1}^{\infty}$ , precisely state what it means for the series  $\sum_{n=1}^{\infty} a_n$  to diverge to  $+\infty$ . (3 marks)
- (ii) Let  $(a_n)_{n=1}^{\infty}$  be a sequence of real numbers with infinitely many positive terms and infinitely many negative terms. Define  $(a_{n_k})_{k=1}^{\infty}$  to be the subsequence obtained by dropping all negative terms, and let  $(a_{m_k})_{k=1}^{\infty}$  be the subsequence formed by all the negative terms. Suppose that

$$\sum_{k=1}^{\infty} a_{n_k} \text{ diverges to } +\infty \quad \text{and} \quad \sum_{k=1}^{\infty} a_{m_k} \text{ converges.}$$

Let  $(b_n)_{n=1}^{\infty}$  be a rearrangement of  $(a_n)_{n=1}^{\infty}$ , that is there exists a bijection  $f : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$  such that  $b_n = a_{f(n)}$ . Carefully prove that  $\sum_{n=1}^{\infty} b_n$  diverges to  $+\infty$ . (5 marks)

- (b) Suppose  $(a_n)_{n=1}^{\infty}$  is a bounded sequence satisfying  $a_{n+1} \geq a_n - 2^{-n-1}$ . Show  $(a_n)_{n=1}^{\infty}$  converges. (5 marks)
- (c) Define the sequence of non-negative real numbers  $(b_n)_{n=1}^{\infty}$  by setting  $b_1 = 1$  and, for  $n \in \mathbb{N}_{>0}$ , setting  $b_{n+1} = \frac{2 + b_n}{1 + b_n}$ . Prove that  $(b_n)_{n=1}^{\infty}$  converges and identify its limit. (7 marks)

(Total: 20 marks)

3. For each of the following statements, decide whether the statement is true or false. If the statement is true, provide a proof. If the statement is false, provide a counter-example with some justification.

- (a) Let  $(a_n)_{n=1}^{\infty}$  be a sequence of non-negative real numbers with  $\sum_{n=1}^{\infty} \sqrt{a_{n+1}a_n}$  convergent.

**True/False:**  $\sum_{n=1}^{\infty} a_n$  must be convergent. (5 marks)

- (b) Let  $(a_n)_{n=1}^{\infty}$  be a sequence of non-negative real numbers with  $\sum_{n=1}^{\infty} a_n$  convergent.

**True/False:**  $\sum_{n=1}^{\infty} \sqrt{a_{n+1}a_n}$  must be convergent. (5 marks)

- (c) Let  $(a_n)_{n=1}^{\infty}$  be a bounded sequence and  $a \in \mathbb{R}$  that together satisfy the following property: any subsequence of  $(a_n)_{n=1}^{\infty}$  that happens to converge must in fact converge to  $a$ .

**True/False:** The entire sequence  $(a_n)_{n=1}^{\infty}$  must converge. (5 marks)

- (d) Let  $(a_n)_{n=1}^{\infty}$  be a sequence such that, for every fixed  $m \in \mathbb{N}_{>0}$ , the sequence  $b_n = a_{n+m} - a_n$  satisfies  $b_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**True/False:** The sequence  $(a_n)_{n=1}^{\infty}$  must converge. (5 marks)

(Total: 20 marks)

4. (a) (i) Carefully state the definition of  $\lim_{x \rightarrow a} f(x) = L$  using  $\epsilon$  and  $\delta$ . (2 marks)
- (ii) Prove that  $\lim_{x \rightarrow a} f(x) = L$  if and only if  $\lim_{n \rightarrow \infty} f(x_n) = L$  for every sequence such that  $\lim_{n \rightarrow \infty} x_n = a$ . (5 marks)
- (iii) Prove that if  $f(x) = x$  for irrational  $x$  and  $f(x) = -x$  for rational  $x$ , then  $\lim_{x \rightarrow a} f(x)$  does not exist if  $a \neq 0$ . (3 marks)
- (b) Conclude whether the following limits exist or not and provide a short proof of your conclusion:
- (i)  $\lim_{x \rightarrow 0} \cos(1/x) \sin(1/x)$ . (2 marks)
- (ii)  $\lim_{x \rightarrow 0} x \cos(1/x) \sin(1/x)$ . (2 marks)
- (iii)  $\lim_{x \rightarrow 0} x^{-1} \sin(ax)$ , where  $a \neq 0$ . (*You are allowed to use that  $\lim_{x \rightarrow 0} x^{-1} \sin(x) = 1$ .*) (3 marks)
- (c) Assume that  $A_n$  is, for each natural number  $n$ , some finite set of numbers in  $[0, 1]$ , and that  $A_n \cap A_m$  is empty when  $m \neq n$ . Define  $f$  as:

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in A_n, \text{ and} \\ 0 & \text{if } x \text{ is not in } A_n \text{ for any } n. \end{cases}$$

Prove that  $\lim_{x \rightarrow a} f(x) = 0$  for all  $a \in [0, 1]$ . (3 marks)

(Total: 20 marks)

5. (a) Carefully state, using  $\epsilon$  and  $\delta$ , what it means for a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  to NOT be continuous at  $a \in \mathbb{R}$ . (3 marks)
- (b) Conclude whether the following statements about  $f : \mathbb{R} \rightarrow \mathbb{R}$  are true or false. If the statement is true provide a one or two sentence proof. If the statement is false provide a counterexample.
- (i) If  $f$  is continuous, not bounded below, not bounded above, and strictly monotone then it is a bijection. (3 marks)
  - (ii) If  $f$  is continuous,  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$  exist and are both finite, then  $f$  is bounded. (4 marks)
  - (iii) If  $f$  is continuous and  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$  exist and are equal, then  $f$  attains either a global maximum or a global minimum. (4 marks)
  - (iv) If  $f$  is continuous, monotone and bounded from below then it attains a global minimum. (3 marks)
- (c) Let  $f : [0, 1] \rightarrow [0, 1]$  be a continuous function. Show there exists  $x \in [0, 1]$  such that  $f(x) = x$ . (3 marks)

(Total: 20 marks)

6. (a) Let  $f$  be a twice differentiable function satisfying  $f''(x) + f'(x)g(x) - f(x) = 0$  for some function  $g$ . Prove that if  $f$  is 0 at two points, then  $f$  is 0 on the interval between them. Hint: evaluate the equation on a positive maximum or a negative minimum. (4 marks)
- (b) Let  $f : [-1, 1] \rightarrow \mathbb{R}$  be defined as  $f(0) = 1$  and  $f(x) = 0$ , for all  $x \neq 0$ . Prove  $f$  is integrable on  $[-1, 1]$  and compute its integral. (5 marks)
- (c) Suppose  $f$  is continuous on  $[a, b]$  and that  $\int_a^b f(x)g(x) = 0$  for all functions  $g$  continuous on  $[a, b]$  and such that  $g(a) = g(b) = 0$ . Conclude that  $f(x) = 0$ , for all  $x \in [a, b]$ . (6 marks)
- (d) Prove that if  $f$  is continuous then  $\int_0^x f(u)(x-u)du = \int_0^x \left( \int_0^u f(t)dt \right) du$ . Hint: differentiate both sides with respect to  $x$ . (5 marks)

(Total: 20 marks)

# 1. (SOLUTIONS)

- (a) (i) The set  $S$  is countably infinite (1 mark). Note that each  $f \in S$  is eventually constant (1 marks), that is for each  $f$ , there exists some  $N \in \mathbb{N}_{>0}$  (depending on  $f$ ) such that for every  $i \geq N$  one has  $f(i) = f(N)$ . We then have  $S = \bigcup_{n=1}^{\infty} A_n$  where  $A_n = \{f \in S : f(i) = f(n) \text{ for all } i \geq n\}$ . Since the countable union of countable sets is countable, it suffices to show that each  $A_n$  is countable (1 mark). To see this, note that there is an injection from  $A_n$  to  $(\mathbb{N}_{>0})^n$  that maps  $A_n \ni f \mapsto (f(1), \dots, f(n)) \in (\mathbb{N}_{>0})^n$  - since a finite Cartesian product of countable sets is countable, it follows that  $A_n$  must be countable (1 marks).
- (ii) The set  $T$  is uncountable (1 mark). We argue by contradiction using Cantor's diagonal argument (1 marks for identifying a workable strategy for proving  $T$  is uncountable). Suppose  $T$  is countable, then we could list its elements  $T = \{f_i\}_{i=1}^{\infty}$ . Now we define a function  $g : \mathbb{N}_{>0} \rightarrow \mathbb{N}_{>0}$  by setting, for each  $j \in \mathbb{N}_{>0}$ ,

$$g(j) = \max(g(1), \dots, g(j-1), f_j(j)) + 1.$$

Clearly  $g$  is increasing so  $g \in T$ , but  $g(j) > f_j(j)$  for every  $j \in \mathbb{N}_{>0}$  so  $g$  is not an element of our list which gives us our contradiction (3 marks for correctly carrying out strategy).

- (b) (i) We first note that for  $n \geq 4$ ,  $n^2 - 5 \geq n^2/2$  (1 mark). Let  $\epsilon > 0$ . We fix  $N = \max(4, 2\lceil 1/\epsilon \rceil)$  (1 marks). It follows that for all  $n \in \mathbb{N}_{>0}$  with  $n \geq N$ ,

$$\left| \frac{1}{n^2 + n - 5} - 0 \right| = \left| \frac{1}{n^2 + n - 5} \right| \leq \frac{2}{n^2} \leq \frac{2}{N^2} \leq \frac{2}{N} < \epsilon.$$

(2 marks)

- (ii) For the  $\Rightarrow$  direction, suppose by contradiction there exists some  $\epsilon > 0$  such that  $y - \epsilon > a$  for all  $a \in A$ . It then follows that  $y - \epsilon$  is an upper bound for  $A$ . However, this is a contradiction since  $y = \sup(A)$  is the least upper bound, and  $y - \epsilon < y$ . (3 marks)
- For the  $\Leftarrow$  direction, suppose  $x = \sup(A) \neq y$ . Since  $x$  is the least upper bound, one must have  $x \leq y$ . We argue by contradiction and suppose  $x < y$ , then set  $\epsilon = y - x > 0$ . From our assumption there exists  $a \in A$  with  $a > y - \epsilon = x$ , but this contradicts  $x$  being an upper bound for  $A$ . (3 marks)

(Total: 0 marks)

## 2. (SOLUTIONS)

- (a) (i)  $\sum_{i=1}^{\infty} a_i$  diverges to  $+\infty$  if, for every  $M > 0$ , there exists  $N \in \mathbb{N}_{>0}$  such that for every  $K \in \mathbb{N}_{>0}$  with  $K \geq N$ ,  $\sum_{n=1}^K a_n > M$ . (1 mark for using finite sums, 2 marks for rest).  
(ii) Let  $M > 0$ . Write  $A = \sum_{k=1}^{\infty} a_{m_k} < 0$ . Since  $\sum_{k=1}^{\infty} a_{n_k}$  diverges to  $+\infty$ , there exists some  $N_1$  such that (1 mark for using assumption of divergence to infinity usefully)

$$\sum_{k=1}^{N_1} a_{n_k} > M - A.$$

Since  $f$  is a bijection, there exists some  $N$  such that  $\{f(1), \dots, f(N)\} \supset \{n_1, \dots, n_{N_1}\}$  (2 marks for correctly using that  $f$  is a bijection).

Next we claim that, for any  $K \geq N$ , we have

$$\sum_{n=1}^K b_n = \sum_{\substack{1 \leq n \leq K \\ b_n \geq 0}} b_n + \sum_{\substack{1 \leq n \leq K \\ b_n < 0}} b_n > M - A + A = M.$$

To see this, note that since  $f$  is a bijection we can find  $K_2 \in \mathbb{N}_{>0}$  such that for all  $\{m_1, \dots, m_{K_2}\} \supset \{f(1), \dots, f(K)\}$ , and so we have (2 marks for rest of argument)

$$\sum_{\substack{1 \leq n \leq K \\ b_n \geq 0}} b_n \geq \sum_{k=1}^{N_1} a_{n_k} > M - A \quad \text{and} \quad \sum_{\substack{1 \leq n \leq K \\ b_n < 0}} b_n \geq \sum_{k=1}^{N_2} a_{m_k} > \sum_{k=1}^{\infty} a_{m_k} = A.$$

- (b) Define the sequence  $(c_n)_{n=1}^{\infty}$  by setting  $c_n = a_n - 2^{-n}$  (2 marks for trying to construct another bounded monotone sequence). Since  $2^{-n}$  is bounded, so is  $(c_n)_{n=1}^{\infty}$ . We also have

$$c_{n+1} - c_n = a_{n+1} - 2^{-n-1} - a_n + 2^{-n} \geq a_n - 2^{-n-1} - 2^{-n-1} - a_n + 2^{-n} = 0.$$

Since the sequence  $(c_n)_{n=1}^{\infty}$  is bounded and increasing it converges (2 marks). Since  $2^{-n} \rightarrow 0$ , by the algebra of limits  $a_n = c_n + 2^{-n}$  also converges (1 mark).

- (c) We rewrite the recursion as  $b_{n+1} = 1 + 1/(b_n + 1)$ . Since  $b_1 \geq 1$ , by induction we have  $b_n \geq 1$  for all  $n \in \mathbb{N}_{>0}$  (1 mark for lower bound). Next we argue by induction that  $|b_{n+1} - b_n| \leq 2^{-n}$ . The base case is given by  $b_2 - b_1 = 3/2 - 1 = 1/2$ . For the inductive step we have

$$|b_{n+2} - b_{n+1}| = \left| \frac{1}{(b_{n+1} + 1)} - \frac{1}{(b_n + 1)} \right| = \frac{|b_{n+1} - b_n|}{(1 + b_{n+1})(1 + b_n)} \leq 2^{-n}/2 = 2^{-n-1}.$$

where in the inequality we used the inductive hypothesis to estimate  $|b_{n+1} - b_n|$  and bounded the denominator from below by 2 since  $b_n, b_{n+1} \geq 1$  (2 marks for bounding differences).

Writing  $b_n = b_1 + \sum_{j=1}^{n-1} (b_{j+1} - b_j)$ , we see that  $(b_n)_{n=1}^{\infty}$  converges if the series  $\sum_{j=1}^{\infty} (b_{j+1} - b_j)$

does. Since  $|b_{j+1} - b_j| \leq 2^{-j}$  it follows that  $\sum_{j=1}^{\infty} (b_{j+1} - b_j)$  is absolutely convergent by

comparison with the convergent geometric series  $\sum_{j=1}^{\infty} 2^{-j}$  (3 marks for comparing to series).

Finally, if  $b_n \rightarrow L$  we have by the algebra of limits that  $L = (2 + L)/(1 + L)$ , so  $L^2 - 2 = 0$ . Since  $L \geq 0$ , it follows that  $L = \sqrt{2}$ . (1 mark for identifying limit)

(Total: 0 marks)

### 3. (SOLUTIONS)

- (a) **False** (1 mark). As a counterexample, let  $a_n = 4^{-n}$  for  $n$  even and  $a_n = 1$  for  $n$  odd (3 marks for example). Then  $\sqrt{a_{n+1}a_n} \leq 2^{-n}$  so  $\sum_{n=1}^{\infty} \sqrt{a_{n+1}a_n}$  is convergent thanks to comparison with the convergent geometric series  $\sum_{n=1}^{\infty} 2^{-n}$ . However  $\sum_{n=1}^{\infty} a_n$  does not converge since  $a_n \not\rightarrow 0$  (1 mark for some justification).
- (b) **True** (1 mark). Using the inequality  $ab \leq \frac{1}{2}(a^2 + b^2)$ , we have that  $\sqrt{a_{n+1}a_n} \leq a_{n+1} + a_n$  (2 marks for using inequality like this). By the algebra of limits for series  $\sum_{n=1}^{\infty} a_n + a_{n+1}$  is convergent, so by comparison the same holds for  $\sum_{n=1}^{\infty} \sqrt{a_{n+1}a_n}$  (2 marks).
- (c) **True** (1 mark). We argue by contradiction and suppose that  $a_n$  does not converge to  $a$ . Then there exists  $\epsilon > 0$  and a subsequence  $a_{n_k}$  such that  $|a_{n_k} - a| > \epsilon$  for all  $k$ . However,  $a_{n_k}$  is a bounded sequence and so itself have a subsequence  $a_{n_{k_j}}$  that converges by the Bolzano-Weierstrass theorem (2 marks for using BW). However,  $a_{n_{k_j}}$  is also a subsequence of  $a_n$ , so we must have  $a_{n_{k_j}} \rightarrow a$  as  $j \rightarrow \infty$ , but this is a contradiction since we always have  $|a_{n_{k_j}} - a| > \epsilon$ . (2 marks for rest of argument).
- (d) **False** (1 mark). For a counterexample, take  $a_n = \sum_{j=1}^n \frac{1}{j}$  (2 marks for counterexample that works). Since the harmonic series diverges, we know that  $(a_n)_{n=1}^{\infty}$  does not converge. At the same time, for any fixed  $m$ ,

$$|b_n| = |a_{m+n} - a_n| = \frac{1}{n+1} + \cdots + \frac{1}{n+m} \leq \frac{m}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(2 marks for justification).

(Total: 0 marks)

#### 4. (SOLUTIONS)

- (a) (i) For every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $x \in (a - \delta, a + \delta) \implies f(x) \in (L - \epsilon, L + \epsilon)$ .
- (ii) ( $\Leftarrow$ , 2 marks) If not, then there exists  $\epsilon > 0$ , such that for all  $n \in \mathbb{N}_{>0}$  there exists  $x_n$  with  $|x_n - a| < 1/n$  and  $|f(x_n) - L| > \epsilon$ . This contradicts  $x_n \rightarrow a \implies f(x_n) \rightarrow L$ . ( $\Rightarrow$ , 3 marks) Choose  $\epsilon > 0$ . Let  $\delta > 0$  be such that  $|x - a| < \delta \implies |f(x) - L| < \epsilon$ . Let  $N \in \mathbb{N}_{>0}$ , be such that  $|x_n - a| < \delta$ , for all  $n \geq N$ . Then,  $|f(x_n) - L| < \epsilon$ , for all  $n \geq N$ .
- (iii) Given  $a \in \mathbb{R}$ , there are sequences  $\{q_n\} \subset \mathbb{Q}$  and  $\{\alpha_n\} \subset \mathbb{R} \setminus \mathbb{Q}$ , such that  $q_n \rightarrow a$  and  $\alpha_n \rightarrow a$ , by the density of rational and irrational numbers. On the other hand  $f(q_n) \rightarrow a$  and  $f(\alpha_n) \rightarrow -a$  which are different if  $a \neq 0$ .
- (b) (i) Does not exist (1 mark). Let  $f(x) = \cos(1/x) \sin(1/x)$ . The sequences  $x_n = (\pi/4 + 2\pi n)^{-1}$  and  $y_n = (5\pi/4 + 2\pi n)^{-1}$  converge to 0, but  $f(x_n)$  is constant and positive, while  $f(y_n)$  is constant and negative (1 mark).
- (ii) Exists (1 mark). Note that  $-x \leq x \cos(1/x) \sin(1/x) \leq x$ , so the limit is zero by the squeeze theorem (1 mark).
- (iii) Exists (1 mark). Given  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|ax| < \delta$ , implies  $|(ax)^{-1} \sin(ax) - 1| < \epsilon|a|^{-1}$ . It follows that if  $|x| < \delta|a|^{-1}$  then  $|x^{-1} \sin(ax) - a| < \epsilon$ . This shows the limit is  $a$  (2 marks).
- (c) Take any sequence  $x_k$  converging to  $a$ . Given  $\epsilon > 0$ , let  $N \in \mathbb{N}_{>0}$  be such that  $1/N < \epsilon$ . The set  $A_1 \cup \dots \cup A_N$  is finite, in particular, there exists  $K \in \mathbb{N}_{>0}$  such that  $k \geq K$  implies that  $x_k$  is not in  $A_1 \cup \dots \cup A_N$ . In particular, for  $k \geq K$ ,  $|f(x_k)| < 1/N < \epsilon$ . This shows  $\lim_{k \rightarrow \infty} f(x_k) = 0$ . Since  $x_k$  was an arbitrary sequence converging to  $a$ , this implies  $\lim_{x \rightarrow a} f(x) = 0$  (1 mark for realising they only have to worry about a finite set + 2 marks for formalising it).

(Total: 0 marks)

## 5. (SOLUTIONS)

- (a) There exists  $\epsilon > 0$ , such that for all  $\delta > 0$ , there exists  $x \in (a-\delta, a+\delta)$  with  $|f(x)-f(a)| > \epsilon$ .
- (b) (i) True (1 mark). Strict monotonicity directly implies injectivity (1 mark). Since  $f$  is neither bounded below nor bounded above, given any  $y \in \mathbb{R}$ , there are  $a, b \in \mathbb{R}$  such that  $f(a) < y < f(b)$ . Surjectivity then follows from the Intermediate Value Theorem because  $f$  is continuous. (1 mark)
- (ii) True (1 mark). Since the limits exist, it follows that  $f$  is bounded on  $(-\infty, R) \cup (R, +\infty)$  for large values of  $R$  (1 mark). On the other hand, since  $f$  is continuous, it is also bounded on  $[-R, R]$  (2 marks).
- (iii) True (1 mark). Let  $L$  be the limit at infinity. If  $f(x) > L$  for some  $x$  then there is a large  $R$  such that  $f(y) < f(x)$  for  $y \in (-\infty, R) \cup (R, +\infty)$  (1 marks). Since  $f$  is continuous it attains a maximum on  $[-R, R]$ , which must be at least  $f(x)$  and therefore is a global maximum. Similarly, if  $f(x) < L$  for some  $x$ , then  $f$  must attain a global minimum. Otherwise  $f(x) = L$ , and all the points are maxima (2 marks).
- (iv) False (1 mark). Consider  $f(x) = e^x$ . (2 marks)
- (c) We can assume  $f(0) \neq 0$  and  $f(1) \neq 1$  as otherwise we would be done. Since  $f([0, 1]) \subset [0, 1]$ , this is the same as assuming  $h(0) > 0$  and  $h(1) < 0$ , where  $h(x) = f(x) - x$  is continuous. By applying the Intermediate Value Theorem to  $h$  it follows that there exists  $x \in [0, 1]$  such that  $h(x) = 0 \implies f(x) = x$ . (1 mark if realising the IVT is the right tool to use + 2 marks for formalising it).

(Total: 0 marks)

## 6. (SOLUTIONS)

- (a) Let  $a < x < b$  with  $f(a) = f(b) = 0$ . Since  $f$  is twice differentiable, then it is also continuous. By the extreme value theorem, there is  $x_0 \in [a, b]$ , such that  $f(x_0) \geq f(x)$ , for all  $x \in [a, b]$ . To proceed by contradiction, assume that  $f(x_0) > 0$ . Therefore,  $x_0$  is an interior maximum (1 mark for realising the need of an interior critical point), so we must have  $f'(x_0) = 0$  and  $f''(x_0) \leq 0$  (2 marks for stating this inequalities). Substituting these into the equation, it follows that  $f(x_0) < 0$ , which contradicts our assumption. Therefore,  $0 \geq f(x)$ , for all  $x \in [a, b]$ . Note that the function  $x \mapsto -f(x)$  satisfies the same equation, so the same argument shows that  $0 \geq -f(x)$ ,  $x \in [a, b]$ . We conclude that  $f = 0$ . (1 mark for formalising the rest).
- (b) For the integrability, it is enough to show that for all  $\epsilon > 0$ , there is a partition  $P$  of  $[-1, 1]$  such that  $U(f, P) - L(f, P) < \epsilon$  (3 marks for stating this inequality). For each  $n \in \mathbb{N}_{>0}$ , let  $P_n$  be the partition  $-1 < -1/n < 1/n < 1$ . Then  $U(f, P_n) = \frac{2}{n}$  and  $L(f, P_n) = 0$ , so it is enough to choose  $n > 2\epsilon^{-1}$ . Moreover, since both  $\lim_{n \rightarrow \infty} U(f, P_n) = \lim_{n \rightarrow \infty} L(f, P_n) = 0$  and  $L(f, P_n) \leq \int_{-1}^1 f(x) dx \leq U(f, P_n)$ , we conclude that  $\int_{-1}^1 f(x) dx = 0$ . from the squeeze theorem. (2 marks if they find a good partition)
- (c) Let  $a < y < b$ . To proceed by contradiction assume that  $f(y) > 0$  (the case  $f(y) < 0$  is similar). Since  $f$  is continuous, there exists  $\delta > 0$  such that  $f(x) > f(y)/2 > 0$ , for all  $x \in [y - 2\delta, y + 2\delta]$  (2 marks). Let  $g$  be a continuous function such that  $g(x) \geq 0$  for all  $x \in [a, b]$  and  $g(x) \geq 1/2$  for all  $x \in [y - \delta, y + \delta]$  (2 marks for this, regardless of whether they give an explicit function or not) (for example,  $g(x) = \max\{1 - |\frac{x-y}{2\delta}|, 0\}$ ). Then  $\int_a^b f(x)g(x)dx \geq \int_{y-\delta}^{y+\delta} f(x)g(x)dx \geq 2\delta \cdot \frac{f(y)}{2} \cdot \frac{1}{2} = \frac{\delta f(y)}{2} > 0$ . Contradicting our assumption. Therefore,  $f \leq 0$ . (2 marks for using the monotonicity of the integral).
- (d) First some notation. Let  $F(x) = \int_0^x f(t)dt$ ,  $G(x) = \int_0^x F(u)du$  and  $H(x) = \int_0^x u f(u)du$ . By assumption, both  $f(x)$  and  $xf(x)$  are integrable, since both  $x$  and  $f(x)$  are continuous (1 mark). Therefore, both  $F$  and  $H$  are well-defined and continuous (1 mark). For the same reason  $G$  is also well-defined. By the Fundamental Theorem of Calculus, which we can apply because we checked all the integrands are continuous, we have  $F'(x) = f(x)$ ,  $G'(x) = F(x)$  and  $H'(x) = xf(x)$  (1 mark). By the Leibniz rule and the formulas above,  $\frac{d}{dx}(xF(x) - H(x) - G(x)) = F(x) + xf(x) - xf(x) - F(x) = 0$ . It follows that  $xF(x) - H(x) - G(x) = 0 \cdot F(0) - H(0) - G(0) = 0$  is the constant zero function (2 marks).

(Total: 0 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

| ExamModuleCode | QuestionNumber | Comments for Students                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
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| MATH40002      | 1              | <p>1)a)i) Many people did well, but many people made some mistakes as well. A small mistake - a lot of people incorrectly assumed the functions were strictly decreasing (that is <math>i &lt; j</math> implies <math>f(j) &lt; f(i)</math>). A bigger mistake was thinking that, given a finite set <math>A</math>, the set of functions going from <math>\mathbb{N}</math> to <math>A</math> is countably infinite - which is not true as soon as <math>A</math> has more than two elements! 1)a)ii) Here, most people who wrote full arguments had the right ideas. A common mistake was thinking the set of all functions can be decomposed as the union of increasing functions with the decreasing functions - but there are many functions that are neither increasing nor decreasing. Another issue is that people used the diagonal argument well but then forgot to ensure that the function they constructed was indeed increasing. Another mistake was trying to list <math>T</math> as functions <math>f_i</math> in a way where <math>f_i(n) \leq f_j(n)</math> for every <math>i \leq j</math> and every <math>n</math> - this is not possible. 1)b)i) Most students did well, but several students tried fixing a numeric value for <math>\epsilon</math> which you shouldn't do in a proof of convergence that has to work for all <math>\epsilon</math>. The most common issue was assuming the sequence was monotone decreasing which is not quite true, you need to enforce <math>N</math> is sufficiently large for this. 1)b)ii) Students for the most part did well on this problem.</p> |
| MATH40002      | 2              | <p>Question 2a i): Most students wrote the correct definition of divergence. Among the students that had it wrong, the most common mistake was to say that the sum diverges to infinity only if the sequence being summed does not tend to 0 or tends to infinity. Question 2a ii): This question had quite a few wrong answers. Most people did not work with partial sums first, passing directly to the infinite sum limit, and justified the result by simply stating that <math>f</math> is a bijection. Question 2b: This question was not done correctly by most students. The most common answers either argued that <math>a_n</math> was increasing, or assumed that the inequality in the question also worked with absolute values to deduce that the sequence was Cauchy. Neither of these hold. Question 2c: This proved to be a challenging question for most students. There were very few correct answers. Most students correctly identified that the limit of the sequence was <math>\sqrt{2}</math>, and pointed out that the sequence was bounded. However, most people then tried to prove that the sequence was monotonic, which is not the case as it oscillates above and below <math>\sqrt{2}</math>.</p>                                                                                                                                                                                                                                                                                                                                                                              |
| MATH40002      | 3              | No Comments Received                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
| MATH40002      | 4              | No Comments Received                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
| MATH40002      | 5              | <p>Part a) A large amount of students stated the correct statement for this and were awarded full marks. Many students tried to attain the result by negating the converse of this statement, yet few did this properly. Part b) Overall students did well in knowing which statements were true or false. Most students struggled with the justification of part two and three. Part c) Most student knew IVT was the correct tool, and the question was generally answered correctly.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |

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| MATH40002 | 6 | <p>a) The majority of students grasped the aim of this question well. A mark was usually lost for lack of rigour. b) Students either found a partition and the corresponding U,P, or stated the desired inequality for integrability, but only a few did both correctly. c) The majority of students struggled with this question. However, some knew the correct approach and did a good job. d) Few students answered this correctly. Rarely did students perform the differentiation of the integrals correctly, nor discuss whether the integrals were well defined or continuous. Unfortunately, no-one stated the use of the Leibniz rule. Some stated the Fundamental Theorem of Calculus and were awarded a mark for this.</p> |
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