

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024**

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Analysis 1

Date: Thursday, May 16, 2024

Time: 14:00 – 17:00 (BST)

Time Allowed: 3 hours

This paper has 6 Questions.

Please Answer Each Question in a Separate Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. For each of the following statements, decide whether the statement is true or false. If the statement is true, provide a proof. If the statement is false, provide a counter-example with some justification.

- (a) Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences with the property that $a_n \rightarrow 0$ and $\sum_{n=1}^{\infty} b_n$ is a convergent series.

True/False: The series $\sum_{n=1}^{\infty} a_n b_n$ must be convergent. (5 marks)

- (b) Let A be a non-empty set and B be a non-empty, finite set. Let S be the set of all functions $f : A \rightarrow B$

True/False: S cannot be countably infinite. (5 marks)

- (c) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence with $(a_{n+1} - a_n) \rightarrow 0$ as $n \rightarrow \infty$.

True/False: The sequence $(a_n)_{n=1}^{\infty}$ must be Cauchy. (5 marks)

- (d) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence such that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

True/False: The series $\sum_{n=1}^{\infty} (a_n)^n$ must converge. (5 marks)

(Total: 20 marks)

2. (a) (i) Precisely state the definition of " $(a_n)_{n=0}^{\infty}$ converges to the real number a ". (3 marks)

- (ii) Prove, from first principles (using only the definition of convergence, and **not** the algebra of limits), that if $a_n \rightarrow a \neq 0$, then $\frac{1}{a_n} \rightarrow \frac{1}{a}$. (6 marks)

- (b) Let A, B be non-empty, bounded sets of positive real numbers. Define $AB = \{ab : a \in A, b \in B\}$. Prove that

$$\sup(AB) = \sup(A) \sup(B).$$

(5 marks)

- (c) Let $(a_n)_{n=0}^{\infty}$ be a sequence with the property that $|a_n|$ does **not** diverge to $+\infty$. Carefully prove that $(a_n)_{n=0}^{\infty}$ has a convergent subsequence. (6 marks)

(Total: 20 marks)

3. (a) Given a sequence $(a_n)_{n=1}^{\infty}$, define the sequence $(b_n)_{n=1}^{\infty}$ by setting $b_n = \frac{1}{n} \sum_{j=1}^n a_j$
- (i) Show that if $a_n \rightarrow 0$, then one must have $b_n \rightarrow 0$. (6 marks)
 - (ii) Give an example sequence $(a_n)_{n=1}^{\infty}$ which does not converge but for which the corresponding $(b_n)_{n=1}^{\infty}$ does converge. (2 marks)
 - (iii) Suppose the sequence $(a_n)_{n=1}^{\infty}$ satisfies $n(a_{n+1} - a_n) \rightarrow 0$, then show that $b_n - a_n \rightarrow 0$. (6 marks)
- (b) Let b be a positive real number, and define the real sequence $(a_n)_{n=1}^{\infty}$ by setting $a_1 = 0$ and enforcing $a_{n+1} = b + a_n^2$. State (with a proof) necessary and sufficient conditions for $(a_n)_{n=1}^{\infty}$ to converge. (6 marks)

(Total: 20 marks)

4. (a) Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, carefully state the formal definition of the following concepts, always using ε and δ :
- (i) $\lim_{x \rightarrow a} f(x) = L$ (2 marks)
 - (ii) f is NOT continuous at a (3 marks)
 - (iii) $\lim_{x \rightarrow +\infty} f(x) \neq L$. (3 marks)
 - (iv) f is uniformly continuous on $X \subset \mathbb{R}$ (3 marks)
- (b) Recall the definition of sequential continuity: we say that f is *sequentially continuous* at a point a , if for every sequence of real numbers $\{x_n\}_{n \in \mathbb{N}}$, $\lim_{n \rightarrow +\infty} x_n = a$ implies $\lim_{n \rightarrow +\infty} f(x_n) = f(a)$.
- (i) Show f continuous at $a \Rightarrow f$ sequentially continuous at a . (2 marks)
 - (ii) Prove, using the result from the previous item, that the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is discontinuous everywhere. (You may use the density of the irrational and rational numbers without proof). (3 marks)

- (iii) Show that if f is sequentially continuous at a then it is also continuous at a . (4 marks)

(Total: 20 marks)

5. (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Determine whether each of the items below is TRUE or FALSE. If TRUE, give a short explanation (one or two lines only). If FALSE, give a counter-example with some explanation (drawings are allowed as counter-examples if accompanied with a clear explanation).
- (i) If f is continuous everywhere, then it is bounded on $\mathbb{R}$. (2 marks)
 - (ii) If f is bounded, then it is uniformly continuous on $\mathbb{R}$. (3 marks)
 - (iii) If f is uniformly continuous, then it is bounded on $\mathbb{R}$. (3 marks)
 - (iv) If f is differentiable everywhere and f' is bounded, then f is uniformly continuous on $\mathbb{R}$. (2 marks)
- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. Assume that the limits $\lim_{x \rightarrow -\infty} f(x) = A$ and $\lim_{x \rightarrow +\infty} f(x) = B$ exist, for $A, B \in \mathbb{R}$.
- (i) Prove that f is bounded. (4 marks)
 - (ii) Give, without proof, an example of a differentiable function f such that the limits above exist and f does not admit any critical points. (3 marks)
 - (iii) Show that if $A = B$, then there exists $x \in \mathbb{R}$ such that $f'(x) = 0$. (3 marks)

(Total: 20 marks)

6. (a) Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Fix $t_0 \in (a, b)$ and define a new function $g : [a, b] \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq t_0 \\ f(t_0) + 1 & \text{if } x = t_0. \end{cases}$$

Prove from first principles that g is integrable on $[a, b]$ (that is, you must use partitions, lower and upper sums). (4 marks)

- (b) Let $f : [0, 1] \rightarrow \mathbb{R}$ be monotone non-decreasing and $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$, for $n \in \mathbb{N}$.
- (i) Calculate the upper and lower sums $U(f, P_n)$ and $L(f, P_n)$. (4 marks)
 - (ii) Define what it means for f to be integrable in $[0, 1]$. (4 marks)
 - (iii) Show that $U(f, P_n) - L(f, P_n) \leq \frac{1}{n}(f(1) - f(0))$. (3 marks)
 - (iv) Conclude that all monotone functions are integrable on closed intervals. (5 marks)

(Total: 20 marks)

1. (SOLUTIONS)

- (a) The statement is false (1 mark). For a counter-example, let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$ (2 marks for counter-example that works) Then $\sum_{n=1}^{\infty} b_n$ converges by the alternating series test, while $a_n \rightarrow 0$, but $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series which is divergent. (2 marks for justification)
- (b) The statement is true (1 mark). If A is finite, then clearly $|S| = |B|^{|A|}$. If $|B| = 1$, then $|S| = 1$. The last case is when $|B| > 1$ and A is infinite. Note that by choosing elements of A , we know there exists a countable subset $\bar{A} = \{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n, \dots\} \subset A$. We also know that there exist distinct $b_1, b_2 \in B$. We now claim that the set

$$\bar{S} = \left\{ f : A \rightarrow B : f(A) \subset \{b_1, b_2\} \text{ and } f(A \setminus \bar{A}) \subset \{b_1\} \right\}$$

is uncountably infinite. Note that since $\bar{S} \subset S$, this shows S is uncountably infinite. We do this by Cantor's diagonal construction (2 marks for having a reasonable strategy for this case). Suppose by contradiction that we have an enumeration $\bar{S} = \{f_1, f_2, \dots, f_n, \dots\}$. We then define $f \in S$ by setting $f(a) = b_1$ for $a \in A \setminus \bar{A}$ and for $\bar{a}_n \in A$,

$$f(\bar{a}_n) = \begin{cases} b_1 & \text{if } f_n(\bar{a}_n) = b_2, \\ b_2 & \text{if } f_n(\bar{a}_n) = b_1. \end{cases}$$

It is easy to see $f \in \bar{S}$, and we also see that for every n , $f(\bar{a}_n) \neq f_n(\bar{a}_n)$ so $f \neq f_n$ (2 marks for finishing proof)

- (c) The statement is false (1 mark). For a counter-example, take $a_n = \sum_{j=1}^n \frac{1}{j}$ (3 marks for counter-example that works). Since a_n is the sequence of partial sums of the harmonic series, it diverges to infinite. However, $a_{n+1} - a_n = \frac{1}{n+1} \rightarrow 0$. (1 mark for justification).
- (d) The statement is true (1 mark). By the divergence test, it follows that $a_n \rightarrow 0$, so there exists some N such that $n \geq N \Rightarrow |a_n| \leq 1$ (2 marks for using terms go to 0). Now note that it suffices to prove that $\sum_{n=N}^{\infty} (a_n)^n$ is absolutely convergent since a series is convergent if its tail is. However, we note that $|a_n^n| \leq |a_n|$ for $n \geq N$, so by comparison with $\sum_{n=N}^{\infty} |a_n|$ we see that $\sum_{n=N}^{\infty} (a_n)^n$ is indeed absolutely convergent (2 marks for rest of argument).

(Total: 0 marks)

2. (SOLUTIONS)

- (a) (i) For any $\epsilon > 0$ (1 mark), there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ (1 mark), $|a_n - a| < \epsilon$ (1 mark)
- (ii) Let $\epsilon_1 = |a|/2$, then there exists N_1 such that for all $n \geq N$, one has $|a_n - a| < \tilde{\epsilon}$. It follows that for $n \geq N_1$, $||a_n| - |a|| \leq |a_n - a| < |a|/2$ which implies

$$|a_n| > |a| - |a|/2 = |a|/2.$$

(3 marks for lower bound)

Now, let $\epsilon > 0$ be arbitrary. It follows that there exists N_2 such that for all $n \geq N_2$, $|a_n - a| < |a|^2\epsilon/2$.

Let $N = \max(N_1, N_2)$, then for all $n \geq N$ one has

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \frac{|a_n - a|}{|a_n| \cdot |a|} < \frac{2}{|a|^2} |a_n - a| < \frac{2}{|a|^2} \cdot |a|^2\epsilon/2 = \epsilon,$$

where in the first inequality above we used that $|a_n| > |a|/2$ since $n \geq N_1$, and in the second inequality we used that $|a_n - a| < |a|^2\epsilon/2$ since $n \geq N_2$. (3 marks for finishing argument).

- (b) We first claim that $\sup(A)\sup(B)$ is an upper bound for AB . To see this note that for any $a \in A$ and $b \in B$, we have $a \leq \sup(A)$ and $b \leq \sup(B)$, from which it follows that $ab \leq \sup(A)\sup(B)$. (2 marks)

We now show that $\sup(A)\sup(B)$ is the least upper bound of AB . Note that for every $n \in \mathbb{N}_{>0}$, there exists $a_n \in A$, and $b_n \in B$ such that $\sup(A) - \frac{1}{n} < a_n \leq \sup(A)$ and $\sup(B) - \frac{1}{n} < b_n \leq \sup(B)$. It follows that $a_n \rightarrow a$ and $b_n \rightarrow b$, and moreover by the algebra of limits $a_nb_n \rightarrow \sup(A)\sup(B)$. Let $\epsilon > 0$, then there exists n such that $|a_nb_n - \sup(A)\sup(B)| < \epsilon \Rightarrow a_nb_n > \sup(A)\sup(B) - \epsilon$. Since $a_nb_n \in AB$, it follows that $\sup(A)\sup(B) - \epsilon$ is not an upper bound of AB , since $\epsilon > 0$ was arbitrary it follows that $\sup(A)\sup(B)$ is indeed the least upper bound of AB . (3 marks for least upper bound)

- (c) Since $|a_n|$ does not diverge to $+\infty$, there exists an $M > 0$ such that, for every $N \in \mathbb{N}$, there exists $n > N$ with $|a_n| < M$. (2 marks for negating diverging to infinity correctly).

We argue that $(a_n)_{n=1}^{\infty}$ contains a subsequence bounded by M . First, by taking $N = 1$ it follows that there exists n_1 such that $|a_{n_1}| < M$. Then, by taking $N = n_1$, it follows that there exists $n_2 > n_1$ such that $|a_{n_2}| < M$. Inductively, assuming that $n_1 < n_2 < \dots < n_k$ have already been chosen, we choose $N = n_k$ above to find a $n_{k+1} > n_k$ such that $|a_{n_{k+1}}| < M$. (2 marks for constructing bounded subsequence)

It follows, by Bolzano-Weierstrass theorem, that the bounded sequence $(a_{n_k})_{k=1}^{\infty}$ contains a convergent subsequence, this gives the desired claim since the subsequence of subsequence of $(a_n)_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. (2 marks for using Bolzano-Weierstrass)

(Total: 0 marks)

3. (SOLUTIONS)

- (a) (i) Since $(a_n)_{n=1}^{\infty}$ is convergent, it is bounded by some M . (1 mark for using boundedness)
 Let $\epsilon > 0$. Then there exists N_1 such that for all $n \geq N_1$, $|a_n| < \frac{\epsilon}{2}$. Furthermore, since $N_1 M/k \rightarrow 0$ as $k \rightarrow \infty$, there exists N_2 , depending on both N_1 and ϵ , such that for all $n \geq N_2$, $N_1 M/n < \epsilon/2$. (2 marks for using convergence) It follows that, for any $n \geq N_2$, (2 marks for breaking things up to use bounded and convergence)

$$|b_n| \leq \frac{1}{n} \left(\sum_{j=1}^{N_1} |a_j| + \sum_{j=N_1}^n |a_j| \right) \leq \frac{N_1 M}{n} + \frac{\epsilon}{2} \frac{|\{j \in \mathbb{N} : N_1 \leq j \leq n\}|}{n} < \epsilon/2 + \epsilon/2 = \epsilon.$$

Above, in the first inequality we used the fact that a_n is bounded, and that for all $n \geq N_1$, $|a_n| < \frac{\epsilon}{2}$. In the second inequality, we used that $\frac{N_1 M}{n} \leq \frac{N_1 M}{N_2} < \epsilon/2$ and that

$$\frac{|\{j \in \mathbb{N} : N_1 \leq j \leq n\}|}{n} \leq 1.$$

It follows that $b_n \rightarrow 0$. (1 mark for finishing argument)

- (ii) We take $a_n = (-1)^n$, then a_n does not converge but $b_n \rightarrow 0$. (2 marks)
 (iii) Let $c_1 = a_1$ and for $n > 1$ set $c_n = a_n - a_{n-1}$. It then follows that $a_n = \sum_{j=1}^n c_j$, and

$$b_n = \frac{1}{n} \sum_{j=1}^n \sum_{\ell=1}^j c_{\ell} = \frac{1}{n} \sum_{j=1}^n (n-j+1)c_j.$$

It follows that (3 marks for finding good way to compare a_n and b_n)

$$a_n - b_n = \sum_{j=1}^n c_j - \frac{1}{n} \sum_{j=1}^n (n-j+1)c_j = \frac{1}{n} \sum_{j=1}^n (j-1)c_j.$$

Since $(j-1)c_j = (j-1)(a_j - a_{j-1}) \rightarrow 0$, by 2(a)i (or equivalently, by the Problem Sheet where this was proven) one has that $\frac{1}{n} \sum_{j=1}^n (j-1)c_j \rightarrow 0$ as $n \rightarrow \infty$, so $a_n \rightarrow b_n \rightarrow 0$. (3 marks for finishing proof)

- (b) Necessary and sufficient conditions are $b \in (0, 1/4]$. We first show necessity. Suppose that $a_n \rightarrow a$, then applying the algebra of limits to $a_{n+1} = b + a_n^2$ gives $a = b + a^2$, which forces $a = \frac{1 \pm \sqrt{1-4b}}{2}$, so we must have $b \leq \frac{1}{4}$ in order for $a = b + a^2$ to have a real solution a . This establishes necessity (3 marks)

We now show sufficiency, suppose $b \in (0, 1/4]$. Note that if $a_n \in [0, 1/2]$, then $a_{n+1} \geq 0$ and $a_{n+1} = b + a_n^2 \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Since $a_1 = 0$, it follows by induction that the sequence $(a_n)_{n=1}^{\infty}$ is contained in $[0, 1/2]$, in particular it is bounded.

We also claim it is monotone decreasing, to see this we observe that

$$a_{n+2} - a_{n+1} = a_{n+1}^2 - a_n^2 = (a_{n+1} + a_n)(a_{n+1} - a_n).$$

Since we know the sequence is non-negative, and that in particular $a_2 \geq 0 = a_1$, it follows by induction that the sequence is monotone increasing. We then conclude that $(a_n)_{n=1}^{\infty}$ is monotone increasing and bounded, therefore convergent. (3 marks for sufficiency)

(Total: 0 marks)

4. (SOLUTIONS)

(a) (i) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.
(2 marks) *if the definition is fully correct. A common mistake is to forget $x \neq a$.*

(ii) There exists $\epsilon_0 > 0$ such that, for all $\delta > 0$, there exists $x \in \mathbb{R}$ satisfying both $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon_0$.
(3 marks) *if the definition is fully correct.*

(iii) There exists $\epsilon_0 > 0$ such that, for all $C > 0$, there exists $x \in \mathbb{R}$ satisfying both $x > C$ and $|f(x) - L| \geq \epsilon_0$.
(3 marks) *if the definition is fully correct.*

(v) For all $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in X$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
(3 marks) *if the definition is fully correct.*

(b) (i) Fix $\epsilon > 0$. Since f is continuous at a , there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Suppose a sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfies $x_n \rightarrow a$. Then, there exists $N > 0$ such that $n \geq N$ implies $|x_n - a| < \delta$. In particular, $n \geq N$ implies $|f(x_n) - f(a)| < \epsilon$.
(1 mark) *for using the individual definitions correctly and an additional*
(1 mark) *for a fully correct answer.*

(ii) Take any $x \in \mathbb{R}$. By the density of rational and irrational numbers, there are sequences $q_n \in \mathbb{Q}$ and $\alpha_n \in \mathbb{R} \setminus \mathbb{Q}$, such that $\lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} \alpha_n = x$. It follows that $f(q_n) = 1$ and $f(\alpha_n) = 0$, for all $n \in \mathbb{N}$. As n goes to ∞ , these sequences respectively converge to 1 and 0, because they are constant sequences. This contradicts sequential continuity at a and by the previous item, it contradicts continuity at a .
(3 marks) *for a fully correct answer. Partial marks allowed for incomplete but reasonable ideas.*

(iii) We proceed by contradiction. Assume continuity at a does not hold. Then, there exists $\epsilon > 0$, such that for all $n \in \mathbb{N}$ there exists $x_n \in \mathbb{R}$ satisfying $|x_n - a| < 1/n$ and $|f(x_n) - f(a)| \geq \epsilon_0$. Therefore $x_n \rightarrow a$ but $f(x_n) \not\rightarrow f(a)$, contradicting sequential continuity.
(2 marks) *for realising the need to use contradiction and an additional*
(2 marks) *for a fully correct answer.*

(Total: 20 marks)

5. (SOLUTIONS)

- (a) (i) FALSE. Take $f(x) = x$, this function is unbounded but continuous.

(2 marks) for any correct example.

- (ii) FALSE. Let $g(x) = \max\{1 - 2|x - 1/2|, 0\}$. This function is continuous, its support is contained in $[0, 1]$ and its image is $[0, 1]$. Let $g_n(x) = g(n(x - n))$, for $n \in \mathbb{N}$. These functions are continuous, their supports $[n, n + 1/n]$ are mutually disjoint, and their image is always $[0, 1]$. It follows that $f(x) = \sum_{n=1}^{\infty} g_n(x)$ is continuous, bounded and not uniformly continuous.

(3 marks) even if only a picture with an explanation is provided. Partial marks allowed for reasonable but incomplete ideas.

- (iii) FALSE. Take $f(x) = x$, this uniformly continuous, since $|f(x) - f(y)| = |x - y| < \epsilon$ if $|x - y| < \epsilon$, but it is clearly not bounded.

(3 marks) even if only a picture with an explanation is provided. Partial marks allowed for reasonable but incomplete ideas.

- (iv) TRUE. Let $M > 0$ be an upper bound for $|f|$. By the Mean Value Theorem, we have $|f(y) - f(x)| \leq M|x - y|$. Therefore, $|f(x) - f(y)| < \epsilon$ if $|x - y| < M\epsilon^{-1}$.

(1 mark) for mentioning the Mean Value Theorem and

(1 mark) for a correct answer.

- (b) (i) Let $M = \max\{|A|, |B|\}$. By assumption, there exists $R > 0$ such that $|f(x)| < 2M$ as long as $|x| > R$. Moreover, since f is differentiable everywhere, then it is also continuous everywhere. It follows from the Extreme Value Theorem that there exists $M' > 0$ such that $|f(x)| < M'$ for $x \in [-2R, 2R]$. Combining both we get $|f(x)| \leq \max\{2M, M'\}$ for all $x \in \mathbb{R}$.

(1 mark) if the existence of the limits is used to guarantee boundedness at infinity,

(1 mark) if the Extreme Value Theorem is used, and

(2 marks) for a complete answer with a partial 1 mark if only an idea is indicated.

- (ii) For example, $f(x) = \frac{e^x - 1}{e^x + 1}$. Note that $\lim_{x \rightarrow +\infty} f(x) = 1$ and $\lim_{x \rightarrow -\infty} f(x) = -1$. However $f'(x) = \frac{2e^x}{(e^x + 1)^2} > 0$.

(2 marks) for the right idea even if presented as a drawing with an explanation

(1 mark) additionally, if fully correct.

- (iii) We prove it by contradiction. If $f'(x) \neq 0$, it follows from the Intermediate Value Property for derivatives that $f' > 0$ or $f' < 0$. By considering $f(-x)$ instead of $f(x)$ if necessary, we can assume, without loss of generality that $f' > 0$. This implies that f is strictly monotone increasing. In particular, $A = \lim_{x \rightarrow -\infty} f(x) < \lim_{x \rightarrow +\infty} f(x) = B$, contradicting our assumption of $A = B$.

(1 mark) mark for mentioning at least one of IVP for derivatives, EVP or Rolle's thm.

(2 marks) for a complete answer with a partial 1 mark if only an idea is indicated.

Other solutions are possible but might be longer. For example, a proof similar to that of Rolle's theorem, or showing the conditions to apply Rolle's theorem.

(Total: 20 marks)

6. (SOLUTIONS)

- (a) Fix $\delta > 0$ smaller than $\min\{b - t_0, t_0 - a\}$. By the property of additivity of the domain, f is integrable $[a, t_0 - \delta]$ and $[t_0 + \delta, b]$. Therefore, there are partitions, P' and P'' of $[a, t_0 - \delta]$ and $[t_0 + \delta, b]$, respectively, such that $U(f, P') - L(f, P') < \delta$ and $U(f, P'') - L(f, P'') < \delta$. Since f is integrable on $[a, b]$, then it is bounded on $[a, b]$. In particular, there exists $M > 0$, independent of δ , such that $|f| < M$ on $[a, b]$. Let $P_\delta = \{t_0 - \delta, t_0 + \delta\}$. Then $U(g, P_\delta) - L(g, P_\delta) < (M + 1) \times 2\delta$. Let $P = P' \cup P_\delta \cup P''$. This is a partition of $[a, b]$ and

$$\begin{aligned} & U(g, P) - L(g, P) \\ &= [U(f, P') - L(f, P')] + [U(g, P_\delta) - L(g, P_\delta)] + [U(f, P'') - L(f, P'')] < 2\delta(M + 2). \end{aligned}$$

Finally, $2\delta(M + 2)$ can be made arbitrarily small by choosing $\delta > 0$ small enough. We conclude that f is integrable.

(1 mark) for attempting to find a partition that makes the difference between upper and lower sums small, (1 mark) mark if the right partition is indicated, and (2 marks) for a complete answer with partial 1 mark if the argument is not entirely correct.

- (b) (i) Since f is monotone non-decreasing, it follows that $m_i = \min_{[t_i, t_{i+1}]} f = f(t_i)$ and $M_i = \max_{[t_i, t_{i+1}]} f = f(t_{i+1})$. Therefore, $U(f, P_n) = \sum_{k=1}^n M_{k-1}(t_k - t_{k-1}) = \sum_{k=1}^n f(t_k) \frac{1}{n}$ and $L(f, P_n) = \sum_{k=1}^n m_{k-1}(t_k - t_{k-1}) = \sum_{k=1}^n f(t_{k-1}) \frac{1}{n}$.

(2 marks) for computing using the correct definition and (2 marks) for computing the sums correctly

- (ii) We say that f is integrable if it is bounded, and the supremum of the set of lower sums $\{L(f, P)\}_{P \in \mathcal{P}}$ coincides with the infimum of the set of upper sums $\{U(f, P)\}_{P \in \mathcal{P}}$, where $\mathcal{P}$ is the set of all finite partitions of $[0, 1]$. Another valid answer would be: f is integrable, if for every $\epsilon > 0$ there exists a partition P of $[0, 1]$ such that $U(f, P) - L(f, P) < \epsilon$.

(4 marks) for a fully correct answer. Partial credit allowed.

- (iii) Let $t_k = \frac{k}{n}$, for $k = 0, 1, \dots, n$. Since $U(f, P_n) = \sum_{k=1}^n f(t_k) \frac{1}{n}$ and $L(f, P_n) = \sum_{k=1}^n f(t_{k-1}) \frac{1}{n}$. It follows that $U(f, P_n) - L(f, P_n) = \frac{1}{n} \sum_{k=1}^n [f(t_k) - f(t_{k-1})] = \frac{1}{n} [f(1) - f(0)]$.

(1 mark) for computing each sum correctly, (2 marks) for a complete answer with partial 1 mark if the argument is not entirely correct.

- (iv) If f is monotone on an interval $[a, b]$, we can repeat the computation above with a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$ such that $t_{i+1} - t_i < \delta$ to conclude $U(f, P) - L(f, P) < \delta|f(b) - f(a)|$. Since this upper bound can be made arbitrarily small, we conclude that f is integrable on $[a, b]$.

(2 marks) for the right idea, and (3 marks) for a complete answer.

(Total: 20 marks)

Question Marker's comment

- 1 The question was overall answered well, with most marks being lost in parts a) and b).1a) This part was the one where the most mistakes were made. A good part of the students tried to prove that this statement was true, hence getting 0 marks. Common mistakes were using the product rule and deducing that the sum was convergent because $a_n \rightarrow 0$. Most of the students who correctly answered "False" were able to provide an appropriate counterexample, with a few exceptions (most commonly choosing a non-summable series for b_n).1b) This part also proved relatively challenging. A good part of the students correctly identified the statement as true. The most common mistakes consisted in not writing enough details in the proof (for example, just mentioning Cantor's diagonal argument without any further explanation), or not considering all possible cases for the size of A .1c) Most people got this right. Some students picked a wrong counterexample (most commonly $1/n$) and a few people lost marks as they tried to prove the sequence was Cauchy.1d) This was also done well. The most common mistakes were stating that the series was convergent by the product rule, which is not applicable in this case, or trying to use the ratio test.
- 2 Question 2(a)(i) This question was well answered. The questions ask for a definition.Question 2(a)(ii) This has been generally well answered but there were some mistakes. Students need to clearly mention the values of N and ϵ that they are using. Some students tried to fix an ϵ in their proof but in general this has to be arbitrary or be shown that it's arbitrary. Some students assumed that the sequence was positive but that wasn't given in the question. Question 2(b) Half of this question was answered well but the other half not so well. Many students were able to prove that $\sup(A)\sup(B)$ is an upper bound but they haven't been able to prove why it's the least upper bound or why $\sup(A)\sup(B) = \sup(AB)$. Some tried to start the argument by assuming $\sup(AB) > \sup(A)\sup(B)$ but this is clearly a contradiction right from the start. Question 2(c) Students were able to identify that the question relates to Bolzano-Weierstrass (BW) theorem but they tried to prove the BW theorem which they don't have to. You just have to use it to show/prove the argument in the question by constructing a bounded subsequence and showing that it converges.Overall there were some answers that used handwavy arguments such as 'from lecture notes' or 'from January test' and so on, although these might have been covered in the notes or in previous tests/problem sets it would be better to state these results in the exam.

Question Marker's comment

- 4 In general question 4a was well answered, most students had done the work to learn the definitions (with some unfortunate exceptions). Many, however, struggled with the negation of statements.
 - 4ai: Well answered, but many missed that you don't care about the value of f at a itself, so you need to specify $0 < |x-a| < \delta$
 - 4aii/aiii: Many struggled to negate "for all x such that" to "there exists an x such that"
 - 4aiv: Some problems with the order of "for all" and "there exists here".
 - 4bi: Well answered amongst those that could recall the definitions, although a few attempts unfortunately tried to use contradiction (which only got full marks on one paper).
 - 4bii: Everyone seemed to have some idea for the correct strategy here, but few were able to complete the argument. Many wasted time trying to find explicit sequences of irrational and rational numbers, and most answers were far longer than was really needed.
 - 4biii: If the student realised they had to use contradiction here they tended to do well, however some struggled with the correct negation of the continuity definition.
- 5 It is important to read the question properly; especially for Question 5. (b) (ii) many example functions did not meet the basic conditions of being differentiable and having limits above and below as required by the question because most students focused more on the critical points part. It's also important to verify that the basic conditions for a theorem are all satisfied before you use the theorem (e.g. closed, bounded, etc.). Unless you justify these, it does not immediately follow that the theorem can be used. The phrase 'from lecture notes' does not constitute real mathematics unless it is written what particular piece of mathematics is actually being used. The lecture notes contain everything required for every solution; you should be specific about which fact/result applies to your solution.