

MATH40003 - Linear Algebra and Group Theory
REVISED HISTORICAL COVERAGE MOCK 1
2026 Structure Reinforcement - one controlled current-format paper

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

Instructions. You may use results from earlier parts of a question in later parts without

reproving them. Let F be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

Timed student version - audit notes are kept in the separate answer and calibration files.

Question 1

Let $D : F^5 \rightarrow F^3$ and $G : F^3 \rightarrow F^5$ be defined by

$$D(a, b, c, d, e)^T = (b - a, d - c, e - c)^T, \quad G(s, t, u)^T = (s, -s, t, u, -t - u)^T.$$

- (a) Write down the matrices of D and G in the standard bases. **(4 marks)**
- (b) Find a basis for $\ker D$ and determine its dimension. **(4 marks)**
- (c) Find $\ker G$, determine $\dim(\operatorname{im} G)$, and prove that

$$\operatorname{im} G = \{x \in F^5 : x_1 + x_2 = 0, x_3 + x_4 + x_5 = 0\}.$$

- (4 marks)**
- (d) Assume that 6 is invertible in F . Prove that $\ker D \cap \operatorname{im} G = \{0\}$. **(4 marks)**
- (e) Under the assumption in part (d), deduce that $F^5 = \ker D \oplus \operatorname{im} G$ and find $\operatorname{rk} D$. **(2 marks)**
- (f) Exhibit a non-zero element of $\ker D \cap \operatorname{im} G$ when $2 = 0$ in F , and another when $3 = 0$ in F . **(2 marks)**

Question 2

Let $V = M_2(F)$ with ordered basis $E = (E_{11}, E_{12}, E_{21}, E_{22})$, and let $T : V \rightarrow V$ be $T(A) = A^T$.

(a) Prove that T is linear and find $[T]_E$. **(3 marks)**

(b) Put $V^+ = \ker(T - I)$ and $V^- = \ker(T + I)$. Find bases for V^+ and V^- and determine $V^+ \cap V^-$, distinguishing $2 \neq 0$ and $2 = 0$ in F . **(5 marks)**

(c) Assume 2 is invertible and set

$$B = (E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21}).$$

Find $[T]_B$. Also find the change-of-basis matrix from B -coordinates to E -coordinates. **(4 marks)**

(d) Assume 2 is invertible. Define $P = \frac{1}{2}(I + T)$ and $K = \frac{1}{2}(I - T)$. Prove that P and K are idempotent, identify their images and kernels, and prove $V = \operatorname{im} P \oplus \operatorname{im} K$. **(5 marks)**

(e) Let $N(A) = E_{12}AE_{12}$. Prove that $N^2 = 0$ and find $(I - N)^{-1}$. **(3 marks)**

Question 3

(a) Let

$$A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 5 \end{pmatrix}.$$

(i) Find the eigenvalues of A . **(3 marks)**

(ii) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$. **(4 marks)**

(iii) Find a real matrix B such that $A = BB^T$. **(3 marks)**

(b) Let $C, D \in M_n(\mathbb{R})$, put $Z = C + iD \in M_n(\mathbb{C})$, and define

$$\mathcal{R}(Z) = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

(i) If Z is unitary, prove that $\mathcal{R}(Z)$ is orthogonal. **(3 marks)**

(ii) Prove that $J\mathcal{R}(Z) = \mathcal{R}(Z)J$. **(3 marks)**

(iii) Conversely, if $R \in M_{2n}(\mathbb{R})$ is orthogonal and $JR = RJ$, prove that $R = \mathcal{R}(Z)$ for a unitary $Z \in M_n(\mathbb{C})$. **(4 marks)**

Question 4

Let $\sigma \in S_{10}$ be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 1 & 6 & 7 & 8 & 9 & 2 & 5 & 3 & 10 \end{pmatrix}.$$

(a) Write σ as a product of disjoint cycles. **(4 marks)**

(b) Write σ as a product of transpositions. **(2 marks)**

(c) Compute $\text{sgn}(\sigma)$. **(2 marks)**

(d) Prove that conjugacy is an equivalence relation on an arbitrary group. **(3 marks)**

(e) If $\tau \in S_n$, prove

$$\tau(a_1 a_2 \cdots a_k) \tau^{-1} = (\tau(a_1) \tau(a_2) \cdots \tau(a_k)).$$

(2 marks)

(f) Prove that two permutations in S_n are conjugate if and only if they have the same cycle type. **(5 marks)**

(g) How many permutations in S_{10} have cycle type $[4, 3, 2, 1]$? **(2 marks)**