

MATH40003 - Linear Algebra and Group Theory
REVISED HISTORICAL COVERAGE MOCK 2
2025 Historical Bridge - direct entry and method-chain training

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

Instructions. You may use results from earlier parts of a question in later parts without

reproving them. Let F be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

Timed student version - audit notes are kept in the separate answer and calibration files.

Question 1

Let

$$M = \begin{pmatrix} 1 & a & 0 & c \\ 0 & b & 1 & 0 \\ 0 & 0 & d & 0 \end{pmatrix} \in M_{3,4}(F).$$

- (a) Determine all values of a, b, c, d for which M is in row-echelon form. **(3 marks)**
 - (b) Determine all values for which M is in reduced row-echelon form. **(3 marks)**
 - (c) Determine $\text{rk } M$ and give bases for the row space and column space in every parameter case. **(6 marks)**
 - (d) Let A, B be matrices of compatible sizes. Prove
 - (i) $\ker B \subseteq \ker(AB)$; **(2 marks)**
 - (ii) $\text{row}(AB) \subseteq \text{row}(B)$; **(2 marks)**
 - (iii) $\text{im}(AB) \subseteq \text{im } A$. **(2 marks)**
- Give 2×2 matrices for which $AB = 0$ but $BA \neq 0$. **(2 marks)**

Question 2

Let V be a vector space over F .

- (a) Give an example of subspaces $U_1, U_2 \leq V$ for which $U_1 \cup U_2$ is not a subspace. **(2 marks)**
- (b) Let $U_1 \subseteq U_2 \subseteq \dots$ be an increasing sequence of subspaces. Prove that $\bigcup_{n \geq 1} U_n$ is a subspace. **(5 marks)**
- (c) Define what it means for V to be finite-dimensional. **(1 marks)**
- (d) If V is finite-dimensional, prove that every increasing chain of subspaces eventually stabilises. **(6 marks)**
- (e) In $F[x]$, construct a strictly increasing sequence of subspaces. **(3 marks)**
- (f) Prove the converse of part (d): if every increasing chain of subspaces of V eventually stabilises, then V is finite-dimensional. **(3 marks)**

Question 3

Let $X = \{1, 2, 3, 4\}$ and let V be the set of all functions $X \rightarrow F$, with pointwise addition and scalar multiplication. Define

$$\Phi : V \rightarrow F^4, \quad \Phi(f) = (f(1), f(2), f(3), f(4))^T.$$

Let $r = (1 \ 2 \ 3 \ 4)$ and define $T(f) = f \circ r$.

(a) Prove that V is a vector space, Φ is a linear isomorphism, and identify the zero function. **(6 marks)**

(b) Let $\delta_j(i) = 1$ if $i = j$ and 0 otherwise. Prove that $(\delta_1, \delta_2, \delta_3, \delta_4)$ is a basis of V . **(3 marks)**

(c) Find the matrix of T in this basis. **(4 marks)**

(d) Find $\ker T$, $\operatorname{im} T$, and $\operatorname{rk} T$. **(3 marks)**

(e) Find $\ker(I - T)$ and prove that

$$\operatorname{im}(I - T) = \{(x_1, x_2, x_3, x_4)^T \in F^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

under the coordinate isomorphism Φ . **(4 marks)**

Question 4

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be

$$T(x, y, z)^T = (2x + y, x + 2y, 4z)^T.$$

- (a) Write down the matrix A of T in the standard basis. **(2 marks)**
- (b) Find all eigenspaces of T and an invertible matrix P for which $P^{-1}AP$ is diagonal. **(8 marks)**
- (c) Let C be an $m \times n$ matrix over a field. Prove that $\text{rk } C < k$ if and only if every $k \times k$ submatrix of C has determinant zero. **(6 marks)**
- (d) For

$$C_t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & t \end{pmatrix},$$

determine $\text{rk } C_t$ for every $t \in \mathbb{R}$.

(4 marks)