

MATH40003 - Linear Algebra and Group Theory
REVISED HISTORICAL COVERAGE MOCK 3
2023-2024 Historical Ability - abstract definitions and construction

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

Instructions. You may use results from earlier parts of a question in later parts without

reproving them. Let F be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

Timed student version - audit notes are kept in the separate answer and calibration files.

Question 1

- (a) Prove that $M_n(F)$, with ordinary matrix addition and multiplication, is a ring. You may quote associativity of matrix multiplication if stated. **(4 marks)**
- (b) Show that each of the following is not a ring:
- (i) $\mathbb{Z}$ with $x \oplus y = |x + y|$ and $x \otimes y = |xy|$; **(2 marks)**
 - (ii) $M_2(\mathbb{R})$ with $A \oplus B = A + B$ and $A \otimes B = (\det A)B$. **(2 marks)**
- (c) Let N_3 be the ring (without identity) of strictly upper-triangular 3×3 real matrices. Prove that its centre is $\text{span}\{E_{13}\}$. **(6 marks)**
- (d) Prove that the centre of $M_n(F)$ consists exactly of the scalar matrices λI_n . **(6 marks)**

Question 2

Let V have ordered basis (b_1, b_2, u, w) , and put

$$U = \text{span}\{b_1, b_2, u\}, \quad W = \text{span}\{b_1, b_2, w\}.$$

- (a) Prove that $U \cap W = \text{span}\{b_1, b_2\}$. **(4 marks)**
- (b) Prove that $U + W = V$ and verify the dimension formula for U and W . **(3 marks)**
- (c) Construct a linear map $f : V \rightarrow U \cap W$ with nullity 2 and $f(U \cap W) = U \cap W$. **(4 marks)**
- (d) Prove that no surjective linear map $g : V \rightarrow V$ can satisfy $g(U) \cap g(W) = \{0\}$. **(4 marks)**
- (e) Construct a linear map $h : V \rightarrow U \cap W$ of rank 2 such that $h(U) \cap h(W) = \{0\}$. **(5 marks)**

Question 3

Let $\beta : V \times V \rightarrow \mathbb{R}$ be bilinear. It is called alternating if $\beta(v, v) = 0$ for every v .

- (a) Prove that every alternating bilinear form is skew-symmetric. **(4 marks)**
- (b) Let A be the matrix of β in a basis of V . Prove that, over $\mathbb{R}$, β is alternating if and only if $A^T = -A$. **(5 marks)**
- (c) Classify all alternating bilinear forms on $\mathbb{R}^3$ by matrices, and prove that every non-zero such form has rank 2. **(5 marks)**
- (d) Prove that every alternating bilinear form on $\mathbb{R}^2$ is a scalar multiple of

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Verify that $J \in SL_2(\mathbb{R})$ and that $(v, w) \mapsto v^T J w$ is alternating. **(6 marks)**

Question 4

Let $r = (1\ 2\ 3\ 4\ 5\ 6)$ and $s = (2\ 6)(3\ 5)$ in S_6 , and let

$$D = \{r^k, r^k s : 0 \leq k < 6\}.$$

- (a) Prove that the twelve displayed elements are distinct, prove $sr s = r^{-1}$, and use the subgroup test to prove that $D \leq S_6$. **(7 marks)**
- (b) Prove that $\text{Aut}(G)$ is a group under composition for every group G . **(3 marks)**
- (c) For $g \in G$, define $C_g(x) = gxg^{-1}$. Prove that $g \mapsto C_g$ is a homomorphism $G \rightarrow \text{Aut}(G)$ with kernel $Z(G)$. **(5 marks)**
- (d) Let $[G, G]$ be the subgroup generated by all commutators. Prove that $[G, G]$ is normal in G . **(3 marks)**
- (e) Prove that $G/[G, G]$ is abelian. **(2 marks)**