

MATH40003 - Linear Algebra and Group Theory**REVISED HISTORICAL COVERAGE MOCK 4**

2020-2022 Historical Ability - foundational methods and classical proofs

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

Instructions. You may use results from earlier parts of a question in later parts without

reproving them. Let F be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

Timed student version - audit notes are kept in the separate answer and calibration files.

Question 1

Consider

$$\begin{cases} x + y + 6z = 4, \\ y + 3z = 2, \\ x + y + 12z = 10. \end{cases}$$

- (a) Find all solutions over $\mathbb{R}$. **(2 marks)**
- (b) Find all solutions over $\mathbb{F}_2$. **(3 marks)**
- (c) Find all solutions over $\mathbb{F}_3$. **(3 marks)**
- (d) Let $T : V \rightarrow W$ be linear. Prove that the solution set of $T(v) = w$ is either empty or a translate of $\ker T$. **(6 marks)**
- (e) For a system $Ax = b$ with n unknowns, prove that it is consistent if and only if $\text{rk } A = \text{rk}(A|b)$; when consistent, prove that its solution set has affine dimension $n - \text{rk } A$. **(6 marks)**

Question 2

Let V have basis $(b_1, \dots, b_n)$, with $n \geq 3$.

(a) For $v = \sum \alpha_i b_i$, consider

$$S_0(v) = 0, \quad S_1(v) = b_1, \quad S_2(v) = \alpha_1 b_1 + \alpha_2 b_2, \quad S_4(v) = \alpha_2 b_1 + \alpha_1 b_2.$$

Determine which maps are linear and which are idempotent, with justification. **(6 marks)**

(b) If $T : V \rightarrow V$ is idempotent, prove that $V = \text{im } T \oplus \ker T$. **(5 marks)**

(c) Define $N(b_1) = 0$ and $N(b_i) = b_{i-1}$ for $i \geq 2$. Prove that N is nilpotent of degree n . More generally, prove that the degree of a nilpotent operator is at most $\dim V$. **(5 marks)**

(d) If $N^m = 0$, prove that $I - N$ and $I + N$ are invertible and give explicit inverses. **(4 marks)**

Question 3

Sequences (a_n) and (b_n) satisfy

$$a_{n+1} = a_n + 2b_n, \quad b_{n+1} = a_n.$$

- (a) Write the recurrence in matrix form and find the eigenvalues and eigenspaces of the matrix. **(5 marks)**
- (b) Derive closed formulae for a_n and b_n in terms of a_1, b_1 . **(4 marks)**
- (c) Find a_5, b_5 when $a_1 = b_1 = 1$. **(1 marks)**
- (d) Prove the Pythagoras identity for a finite orthogonal set of non-zero vectors. **(2 marks)**
- (e) Let S be a real symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Prove

$$\max_{\|v\|=1} \|Sv\| = \max_i |\lambda_i|, \quad \min_{\|v\|=1} \|Sv\| = \min_i |\lambda_i|.$$

(8 marks)

Question 4

Let $G = \langle g \rangle$ be cyclic of order 12.

- (a) Determine the possible orders of elements of G and the number of elements of each order. **(5 marks)**

- (b) Find proper subgroups A, B such that $A \cap B = \{e\}$ and $AB = G$, and justify the claims. **(4 marks)**

- (c) If $\phi : G_1 \rightarrow G_2$ is a homomorphism, prove that $\ker \phi$ is a subgroup and that

$$\phi(x) = \phi(y) \iff x \ker \phi = y \ker \phi.$$

(5 marks)

- (d) Let H be a group of prime order and let $\psi : H \rightarrow K$ be a non-trivial homomorphism. Prove that ψ is injective. **(3 marks)**

- (e) Give non-trivial homomorphisms $GL_2(\mathbb{R}) \rightarrow \mathbb{R}^\times$ and $(\mathbb{R}, +) \rightarrow GL_2(\mathbb{R})$, and verify them. **(3 marks)**