

**MATH40003 - Linear Algebra and Group Theory**  
**REVISED HISTORICAL COVERAGE MOCK 5**  
Seven-Year Gap Closure - high-mark and rotating ability gaps

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

**Instructions.** You may use results from earlier parts of a question in later parts without

reproving them. Let  $F$  be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

*Timed student version - audit notes are kept in the separate answer and calibration files.*

## Question 1

Let  $F$  be a field.

- (a) Suppose  $A \in M_n(F)$  has exactly one non-zero entry in each row and each column, with non-zero entries  $x_1, \dots, x_n$ . Prove that  $\det A = \pm x_1 \cdots x_n$ . **(4 marks)**
- (b) One zero entry of  $A$  is changed to a scalar  $\lambda$ , producing  $B$ . Prove that  $\det B = \det A$ . **(4 marks)**
- (c) Let  $C \in M_4(F)$  and suppose the eigenspace  $E_\mu$  has dimension 2. Prove directly that

$$\chi_C(t) = (t - \mu)^2 q(t)$$

for some  $q(t) \in F[t]$ . **(5 marks)**

- (d) Prove that the rank of a matrix is the largest integer  $k$  for which it has a  $k \times k$  submatrix with non-zero determinant. **(7 marks)**

## Question 2

Let  $b, c \in \mathbb{R}$  with  $b \neq c$ . For  $n \geq 1$ , let  $T_n$  be the tridiagonal matrix with diagonal entries  $b + c$ , superdiagonal entries  $b$ , and subdiagonal entries  $c$ . Put  $d_n = \det T_n$  and  $d_0 = 1$ .

(a) Prove

$$d_n = (b + c)d_{n-1} - bc d_{n-2} \quad (n \geq 2).$$

**(5 marks)**

(b) Write this recurrence as a first-order matrix recurrence for  $(d_n, d_{n-1})^T$ . **(3 marks)**

(c) Diagonalise the  $2 \times 2$  recurrence matrix and prove

$$d_n = \frac{b^{n+1} - c^{n+1}}{b - c}.$$

**(8 marks)**

(d) Check the result for  $b = 2, c = 1, n = 4$  by both the recurrence and the closed formula. **(4 marks)**

### Question 3

Let  $A$  have columns

$$v_1 = (1, 1, 0)^T, \quad v_2 = (1, 0, 1)^T, \quad v_3 = (0, 1, 1)^T.$$

- (a) Apply Gram-Schmidt to obtain an orthonormal basis and hence a QR factorisation  $A = QR$  with positive diagonal entries in  $R$ . **(8 marks)**
- (b) Prove that every matrix in  $O_2(\mathbb{R})$  is either a rotation or a reflection. **(5 marks)**
- (c) Prove that  $O_n(\mathbb{Z})$  consists exactly of the signed permutation matrices. Deduce that it is a subgroup of  $O_n(\mathbb{R})$  and that

$$|O_n(\mathbb{Z})| = 2^n n!.$$

**(7 marks)**

## Question 4

- (a) Let  $H = \{g^2 : g \in G\}$ . Prove that  $H$  is a subgroup when  $G$  is abelian, and prove that  $H = G$  when  $G$  is finite of odd order. **(6 marks)**
- (b) Let  $p, q$  be distinct primes. Prove that  $z \mapsto z^q$  is an automorphism of the group of  $p$ th roots of unity. **(4 marks)**
- (c) Determine the possible cycle shapes of elements of order 3 in  $S_8$  and count all such elements. **(4 marks)**
- (d) Prove that the number of elements of order  $p$  in a finite group is divisible by  $p - 1$ . **(3 marks)**
- (e) Prove Cayley's theorem for finite groups. **(3 marks)**