

MATH40003 - Linear Algebra and Group Theory

REVISED HISTORICAL COVERAGE MOCK 6

Seven-Year Mixed Transfer Pressure - unfamiliar wording, familiar course methods

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

Instructions. You may use results from earlier parts of a question in later parts without

reproving them. Let F be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

Timed student version - audit notes are kept in the separate answer and calibration files.

Question 1

Let $X = \{1, 2, 3, 4, 5, 6\}$, let V be the vector space of functions $X \rightarrow F$, and let $\tau = (1\ 2)(3\ 4\ 5)$, fixing 6. Define $T(f) = f \circ \tau$.

- (a) Prove $V \cong F^6$ and give the delta-function basis. **(4 marks)**
- (b) Prove that T is linear and find its matrix in the delta-function basis. **(4 marks)**
- (c) Find a basis for $\ker(I - T)$. **(4 marks)**
- (d) Under the coordinate isomorphism, prove

$$\operatorname{im}(I - T) = \{x \in F^6 : x_1 + x_2 = 0, x_3 + x_4 + x_5 = 0, x_6 = 0\},$$

and find its rank. **(4 marks)**

- (e) Determine $\ker(I - T) \cap \operatorname{im}(I - T)$ when $\operatorname{char} F$ is neither 2 nor 3, when it is 2, and when it is 3. **(4 marks)**

Question 2

Let $A : V \rightarrow V$ be linear and satisfy $A^3 = A$.

(a) Prove that A^2 is idempotent, $\ker A^2 = \ker A$, and $\operatorname{im} A^2 = \operatorname{im} A$. **(4 marks)**

(b) Assume 2 is invertible and define

$$P_0 = I - A^2, \quad P_+ = \frac{A^2 + A}{2}, \quad P_- = \frac{A^2 - A}{2}.$$

Prove that the P 's are pairwise orthogonal idempotents whose sum is I , and identify their images. **(10 marks)**

(c) Deduce that

$$V = \ker A \oplus \ker(A - I) \oplus \ker(A + I)$$

and that A is diagonalisable. **(3 marks)**

(d) Over a field of characteristic 2, give a non-diagonalisable 2×2 matrix satisfying $A^3 = A$, and verify both statements. **(3 marks)**

Question 3

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

- (a) Find an orthogonal diagonalisation of A . **(6 marks)**
- (b) Find B such that $A = BB^T$. **(3 marks)**
- (c) Let S be any real symmetric matrix with eigenvalues λ_i . Prove the maximum and minimum norm formulae on the unit sphere. **(5 marks)**
- (d) Prove that every invertible real matrix has a QR factorisation with an orthogonal first factor and an upper-triangular second factor with positive diagonal. **(6 marks)**

Question 4

(a) In S_6 , simplify

$$(1\ 2\ 3\ 4\ 5\ 6)(1\ 2\ 3\ 4\ 5)(1\ 2\ 3\ 4)(1\ 2\ 3)$$

into disjoint cycles, and determine its order and sign.

(4 marks)

(b) What is the maximum order of an element of S_9 ? Justify your answer.

(4 marks)

(c) Determine whether each group is cyclic, with proof:

$$S_5, \quad (\mathbb{Z} \times \mathbb{Z}, +), \quad \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}, \quad \text{rotations of a regular } m\text{-gon.}$$

(5 marks)

(d) Let every element of a group G have finite order. If $H \subseteq G$ is non-empty and closed under multiplication, prove that H is a subgroup.

(7 marks)