

**MATH40003 - Linear Algebra and Group Theory**  
**REVISED HISTORICAL COVERAGE MOCK 1**  
2026 Structure Reinforcement - one controlled current-format paper

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

**Instructions.** You may use results from earlier parts of a question in later parts without reproving them. Let  $F$  be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

*Timed student version - audit notes are kept in the separate answer and calibration files.*

## Question 1

Let  $D : F^5 \rightarrow F^3$  and  $G : F^3 \rightarrow F^5$  be defined by

$$D(a, b, c, d, e)^T = (b - a, d - c, e - c)^T, \quad G(s, t, u)^T = (s, -s, t, u, -t - u)^T.$$

- (a) Write down the matrices of  $D$  and  $G$  in the standard bases. **(4 marks)**
- (b) Find a basis for  $\ker D$  and determine its dimension. **(4 marks)**
- (c) Find  $\ker G$ , determine  $\dim(\operatorname{im} G)$ , and prove that

$$\operatorname{im} G = \{x \in F^5 : x_1 + x_2 = 0, x_3 + x_4 + x_5 = 0\}.$$

- (4 marks)**
- (d) Assume that 6 is invertible in  $F$ . Prove that  $\ker D \cap \operatorname{im} G = \{0\}$ . **(4 marks)**
- (e) Under the assumption in part (d), deduce that  $F^5 = \ker D \oplus \operatorname{im} G$  and find  $\operatorname{rk} D$ . **(2 marks)**
- (f) Exhibit a non-zero element of  $\ker D \cap \operatorname{im} G$  when  $2 = 0$  in  $F$ , and another when  $3 = 0$  in  $F$ . **(2 marks)**

## Question 2

Let  $V = M_2(F)$  with ordered basis  $E = (E_{11}, E_{12}, E_{21}, E_{22})$ , and let  $T : V \rightarrow V$  be  $T(A) = A^T$ .

(a) Prove that  $T$  is linear and find  $[T]_E$ . **(3 marks)**

(b) Put  $V^+ = \ker(T - I)$  and  $V^- = \ker(T + I)$ . Find bases for  $V^+$  and  $V^-$  and determine  $V^+ \cap V^-$ , distinguishing  $2 \neq 0$  and  $2 = 0$  in  $F$ . **(5 marks)**

(c) Assume  $2$  is invertible and set

$$B = (E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21}).$$

Find  $[T]_B$ . Also find the change-of-basis matrix from  $B$ -coordinates to  $E$ -coordinates. **(4 marks)**

(d) Assume  $2$  is invertible. Define  $P = \frac{1}{2}(I + T)$  and  $K = \frac{1}{2}(I - T)$ . Prove that  $P$  and  $K$  are idempotent, identify their images and kernels, and prove  $V = \operatorname{im} P \oplus \operatorname{im} K$ . **(5 marks)**

(e) Let  $N(A) = E_{12}AE_{12}$ . Prove that  $N^2 = 0$  and find  $(I - N)^{-1}$ . **(3 marks)**

### Question 3

(a) Let

$$A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 5 \end{pmatrix}.$$

(i) Find the eigenvalues of  $A$ . **(3 marks)**

(ii) Find an orthogonal matrix  $Q$  and a diagonal matrix  $\Lambda$  such that  $A = Q\Lambda Q^T$ . **(4 marks)**

(iii) Find a real matrix  $B$  such that  $A = BB^T$ . **(3 marks)**

(b) Let  $C, D \in M_n(\mathbb{R})$ , put  $Z = C + iD \in M_n(\mathbb{C})$ , and define

$$\mathcal{R}(Z) = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

(i) If  $Z$  is unitary, prove that  $\mathcal{R}(Z)$  is orthogonal. **(3 marks)**

(ii) Prove that  $J\mathcal{R}(Z) = \mathcal{R}(Z)J$ . **(3 marks)**

(iii) Conversely, if  $R \in M_{2n}(\mathbb{R})$  is orthogonal and  $JR = RJ$ , prove that  $R = \mathcal{R}(Z)$  for a unitary  $Z \in M_n(\mathbb{C})$ . **(4 marks)**

## Question 4

Let  $\sigma \in S_{10}$  be given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 1 & 6 & 7 & 8 & 9 & 2 & 5 & 3 & 10 \end{pmatrix}.$$

(a) Write  $\sigma$  as a product of disjoint cycles. **(4 marks)**

(b) Write  $\sigma$  as a product of transpositions. **(2 marks)**

(c) Compute  $\text{sgn}(\sigma)$ . **(2 marks)**

(d) Prove that conjugacy is an equivalence relation on an arbitrary group. **(3 marks)**

(e) If  $\tau \in S_n$ , prove

$$\tau(a_1 a_2 \cdots a_k) \tau^{-1} = (\tau(a_1) \tau(a_2) \cdots \tau(a_k)).$$

**(2 marks)**

(f) Prove that two permutations in  $S_n$  are conjugate if and only if they have the same cycle type. **(5 marks)**

(g) How many permutations in  $S_{10}$  have cycle type  $[4, 3, 2, 1]$ ? **(2 marks)**

**MATH40003 - Linear Algebra and Group Theory**  
**REVISED HISTORICAL COVERAGE MOCK 2**  
2025 Historical Bridge - direct entry and method-chain training

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

**Instructions.** You may use results from earlier parts of a question in later parts without

reproving them. Let  $F$  be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

*Timed student version - audit notes are kept in the separate answer and calibration files.*

## Question 1

Let

$$M = \begin{pmatrix} 1 & a & 0 & c \\ 0 & b & 1 & 0 \\ 0 & 0 & d & 0 \end{pmatrix} \in M_{3,4}(F).$$

- (a) Determine all values of  $a, b, c, d$  for which  $M$  is in row-echelon form. **(3 marks)**
  - (b) Determine all values for which  $M$  is in reduced row-echelon form. **(3 marks)**
  - (c) Determine  $\text{rk } M$  and give bases for the row space and column space in every parameter case. **(6 marks)**
  - (d) Let  $A, B$  be matrices of compatible sizes. Prove
    - (i)  $\ker B \subseteq \ker(AB)$ ; **(2 marks)**
    - (ii)  $\text{row}(AB) \subseteq \text{row}(B)$ ; **(2 marks)**
    - (iii)  $\text{im}(AB) \subseteq \text{im } A$ . **(2 marks)**
- Give  $2 \times 2$  matrices for which  $AB = 0$  but  $BA \neq 0$ . **(2 marks)**

## Question 2

Let  $V$  be a vector space over  $F$ .

- (a) Give an example of subspaces  $U_1, U_2 \leq V$  for which  $U_1 \cup U_2$  is not a subspace. **(2 marks)**
- (b) Let  $U_1 \subseteq U_2 \subseteq \dots$  be an increasing sequence of subspaces. Prove that  $\bigcup_{n \geq 1} U_n$  is a subspace. **(5 marks)**
- (c) Define what it means for  $V$  to be finite-dimensional. **(1 marks)**
- (d) If  $V$  is finite-dimensional, prove that every increasing chain of subspaces eventually stabilises. **(6 marks)**
- (e) In  $F[x]$ , construct a strictly increasing sequence of subspaces. **(3 marks)**
- (f) Prove the converse of part (d): if every increasing chain of subspaces of  $V$  eventually stabilises, then  $V$  is finite-dimensional. **(3 marks)**

### Question 3

Let  $X = \{1, 2, 3, 4\}$  and let  $V$  be the set of all functions  $X \rightarrow F$ , with pointwise addition and scalar multiplication. Define

$$\Phi : V \rightarrow F^4, \quad \Phi(f) = (f(1), f(2), f(3), f(4))^T.$$

Let  $r = (1\ 2\ 3\ 4)$  and define  $T(f) = f \circ r$ .

(a) Prove that  $V$  is a vector space,  $\Phi$  is a linear isomorphism, and identify the zero function. **(6 marks)**

(b) Let  $\delta_j(i) = 1$  if  $i = j$  and 0 otherwise. Prove that  $(\delta_1, \delta_2, \delta_3, \delta_4)$  is a basis of  $V$ . **(3 marks)**

(c) Find the matrix of  $T$  in this basis. **(4 marks)**

(d) Find  $\ker T$ ,  $\operatorname{im} T$ , and  $\operatorname{rk} T$ . **(3 marks)**

(e) Find  $\ker(I - T)$  and prove that

$$\operatorname{im}(I - T) = \{(x_1, x_2, x_3, x_4)^T \in F^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

under the coordinate isomorphism  $\Phi$ . **(4 marks)**

## Question 4

Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be

$$T(x, y, z)^T = (2x + y, x + 2y, 4z)^T.$$

- (a) Write down the matrix  $A$  of  $T$  in the standard basis. **(2 marks)**
- (b) Find all eigenspaces of  $T$  and an invertible matrix  $P$  for which  $P^{-1}AP$  is diagonal. **(8 marks)**
- (c) Let  $C$  be an  $m \times n$  matrix over a field. Prove that  $\text{rk } C < k$  if and only if every  $k \times k$  submatrix of  $C$  has determinant zero. **(6 marks)**
- (d) For

$$C_t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & t \end{pmatrix},$$

determine  $\text{rk } C_t$  for every  $t \in \mathbb{R}$ .

**(4 marks)**

**MATH40003 - Linear Algebra and Group Theory**  
**REVISED HISTORICAL COVERAGE MOCK 3**  
2023-2024 Historical Ability - abstract definitions and construction

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

**Instructions.** You may use results from earlier parts of a question in later parts without reproving them. Let  $F$  be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

*Timed student version - audit notes are kept in the separate answer and calibration files.*

## Question 1

- (a) Prove that  $M_n(F)$ , with ordinary matrix addition and multiplication, is a ring. You may quote associativity of matrix multiplication if stated. **(4 marks)**
- (b) Show that each of the following is not a ring:
- (i)  $\mathbb{Z}$  with  $x \oplus y = |x + y|$  and  $x \otimes y = |xy|$ ; **(2 marks)**
  - (ii)  $M_2(\mathbb{R})$  with  $A \oplus B = A + B$  and  $A \otimes B = (\det A)B$ . **(2 marks)**
- (c) Let  $N_3$  be the ring (without identity) of strictly upper-triangular  $3 \times 3$  real matrices. Prove that its centre is  $\text{span}\{E_{13}\}$ . **(6 marks)**
- (d) Prove that the centre of  $M_n(F)$  consists exactly of the scalar matrices  $\lambda I_n$ . **(6 marks)**

## Question 2

Let  $V$  have ordered basis  $(b_1, b_2, u, w)$ , and put

$$U = \text{span}\{b_1, b_2, u\}, \quad W = \text{span}\{b_1, b_2, w\}.$$

- (a) Prove that  $U \cap W = \text{span}\{b_1, b_2\}$ . **(4 marks)**
- (b) Prove that  $U + W = V$  and verify the dimension formula for  $U$  and  $W$ . **(3 marks)**
- (c) Construct a linear map  $f : V \rightarrow U \cap W$  with nullity 2 and  $f(U \cap W) = U \cap W$ . **(4 marks)**
- (d) Prove that no surjective linear map  $g : V \rightarrow V$  can satisfy  $g(U) \cap g(W) = \{0\}$ . **(4 marks)**
- (e) Construct a linear map  $h : V \rightarrow U \cap W$  of rank 2 such that  $h(U) \cap h(W) = \{0\}$ . **(5 marks)**

### Question 3

Let  $\beta : V \times V \rightarrow \mathbb{R}$  be bilinear. It is called alternating if  $\beta(v, v) = 0$  for every  $v$ .

- (a) Prove that every alternating bilinear form is skew-symmetric. **(4 marks)**
- (b) Let  $A$  be the matrix of  $\beta$  in a basis of  $V$ . Prove that, over  $\mathbb{R}$ ,  $\beta$  is alternating if and only if  $A^T = -A$ . **(5 marks)**
- (c) Classify all alternating bilinear forms on  $\mathbb{R}^3$  by matrices, and prove that every non-zero such form has rank 2. **(5 marks)**
- (d) Prove that every alternating bilinear form on  $\mathbb{R}^2$  is a scalar multiple of

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Verify that  $J \in SL_2(\mathbb{R})$  and that  $(v, w) \mapsto v^T J w$  is alternating. **(6 marks)**

## Question 4

Let  $r = (1\ 2\ 3\ 4\ 5\ 6)$  and  $s = (2\ 6)(3\ 5)$  in  $S_6$ , and let

$$D = \{r^k, r^k s : 0 \leq k < 6\}.$$

- (a) Prove that the twelve displayed elements are distinct, prove  $sr s = r^{-1}$ , and use the subgroup test to prove that  $D \leq S_6$ . **(7 marks)**
- (b) Prove that  $\text{Aut}(G)$  is a group under composition for every group  $G$ . **(3 marks)**
- (c) For  $g \in G$ , define  $C_g(x) = gxg^{-1}$ . Prove that  $g \mapsto C_g$  is a homomorphism  $G \rightarrow \text{Aut}(G)$  with kernel  $Z(G)$ . **(5 marks)**
- (d) Let  $[G, G]$  be the subgroup generated by all commutators. Prove that  $[G, G]$  is normal in  $G$ . **(3 marks)**
- (e) Prove that  $G/[G, G]$  is abelian. **(2 marks)**

# **MATH40003 - Linear Algebra and Group Theory**

## **REVISED HISTORICAL COVERAGE MOCK 4**

2020-2022 Historical Ability - foundational methods and classical proofs

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

**Instructions.** You may use results from earlier parts of a question in later parts without

reproving them. Let  $F$  be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

*Timed student version - audit notes are kept in the separate answer and calibration files.*

## Question 1

Consider

$$\begin{cases} x + y + 6z = 4, \\ y + 3z = 2, \\ x + y + 12z = 10. \end{cases}$$

- (a) Find all solutions over  $\mathbb{R}$ . **(2 marks)**
- (b) Find all solutions over  $\mathbb{F}_2$ . **(3 marks)**
- (c) Find all solutions over  $\mathbb{F}_3$ . **(3 marks)**
- (d) Let  $T : V \rightarrow W$  be linear. Prove that the solution set of  $T(v) = w$  is either empty or a translate of  $\ker T$ . **(6 marks)**
- (e) For a system  $Ax = b$  with  $n$  unknowns, prove that it is consistent if and only if  $\text{rk } A = \text{rk}(A|b)$ ; when consistent, prove that its solution set has affine dimension  $n - \text{rk } A$ . **(6 marks)**

## Question 2

Let  $V$  have basis  $(b_1, \dots, b_n)$ , with  $n \geq 3$ .

(a) For  $v = \sum \alpha_i b_i$ , consider

$$S_0(v) = 0, \quad S_1(v) = b_1, \quad S_2(v) = \alpha_1 b_1 + \alpha_2 b_2, \quad S_4(v) = \alpha_2 b_1 + \alpha_1 b_2.$$

Determine which maps are linear and which are idempotent, with justification. **(6 marks)**

(b) If  $T : V \rightarrow V$  is idempotent, prove that  $V = \text{im } T \oplus \ker T$ . **(5 marks)**

(c) Define  $N(b_1) = 0$  and  $N(b_i) = b_{i-1}$  for  $i \geq 2$ . Prove that  $N$  is nilpotent of degree  $n$ . More generally, prove that the degree of a nilpotent operator is at most  $\dim V$ . **(5 marks)**

(d) If  $N^m = 0$ , prove that  $I - N$  and  $I + N$  are invertible and give explicit inverses. **(4 marks)**

### Question 3

Sequences  $(a_n)$  and  $(b_n)$  satisfy

$$a_{n+1} = a_n + 2b_n, \quad b_{n+1} = a_n.$$

- (a) Write the recurrence in matrix form and find the eigenvalues and eigenspaces of the matrix. **(5 marks)**
- (b) Derive closed formulae for  $a_n$  and  $b_n$  in terms of  $a_1, b_1$ . **(4 marks)**
- (c) Find  $a_5, b_5$  when  $a_1 = b_1 = 1$ . **(1 marks)**
- (d) Prove the Pythagoras identity for a finite orthogonal set of non-zero vectors. **(2 marks)**
- (e) Let  $S$  be a real symmetric matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$ . Prove

$$\max_{\|v\|=1} \|Sv\| = \max_i |\lambda_i|, \quad \min_{\|v\|=1} \|Sv\| = \min_i |\lambda_i|.$$

**(8 marks)**

## Question 4

Let  $G = \langle g \rangle$  be cyclic of order 12.

- (a) Determine the possible orders of elements of  $G$  and the number of elements of each order. **(5 marks)**

- (b) Find proper subgroups  $A, B$  such that  $A \cap B = \{e\}$  and  $AB = G$ , and justify the claims. **(4 marks)**

- (c) If  $\phi : G_1 \rightarrow G_2$  is a homomorphism, prove that  $\ker \phi$  is a subgroup and that

$$\phi(x) = \phi(y) \iff x \ker \phi = y \ker \phi.$$

**(5 marks)**

- (d) Let  $H$  be a group of prime order and let  $\psi : H \rightarrow K$  be a non-trivial homomorphism. Prove that  $\psi$  is injective. **(3 marks)**

- (e) Give non-trivial homomorphisms  $GL_2(\mathbb{R}) \rightarrow \mathbb{R}^\times$  and  $(\mathbb{R}, +) \rightarrow GL_2(\mathbb{R})$ , and verify them. **(3 marks)**

**MATH40003 - Linear Algebra and Group Theory**  
**REVISED HISTORICAL COVERAGE MOCK 5**  
Seven-Year Gap Closure - high-mark and rotating ability gaps

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

**Instructions.** You may use results from earlier parts of a question in later parts without

reproving them. Let  $F$  be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

*Timed student version - audit notes are kept in the separate answer and calibration files.*

## Question 1

Let  $F$  be a field.

- (a) Suppose  $A \in M_n(F)$  has exactly one non-zero entry in each row and each column, with non-zero entries  $x_1, \dots, x_n$ . Prove that  $\det A = \pm x_1 \cdots x_n$ . **(4 marks)**
- (b) One zero entry of  $A$  is changed to a scalar  $\lambda$ , producing  $B$ . Prove that  $\det B = \det A$ . **(4 marks)**
- (c) Let  $C \in M_4(F)$  and suppose the eigenspace  $E_\mu$  has dimension 2. Prove directly that

$$\chi_C(t) = (t - \mu)^2 q(t)$$

for some  $q(t) \in F[t]$ . **(5 marks)**

- (d) Prove that the rank of a matrix is the largest integer  $k$  for which it has a  $k \times k$  submatrix with non-zero determinant. **(7 marks)**

## Question 2

Let  $b, c \in \mathbb{R}$  with  $b \neq c$ . For  $n \geq 1$ , let  $T_n$  be the tridiagonal matrix with diagonal entries  $b + c$ , superdiagonal entries  $b$ , and subdiagonal entries  $c$ . Put  $d_n = \det T_n$  and  $d_0 = 1$ .

(a) Prove

$$d_n = (b + c)d_{n-1} - bc d_{n-2} \quad (n \geq 2).$$

**(5 marks)**

(b) Write this recurrence as a first-order matrix recurrence for  $(d_n, d_{n-1})^T$ . **(3 marks)**

(c) Diagonalise the  $2 \times 2$  recurrence matrix and prove

$$d_n = \frac{b^{n+1} - c^{n+1}}{b - c}.$$

**(8 marks)**

(d) Check the result for  $b = 2, c = 1, n = 4$  by both the recurrence and the closed formula. **(4 marks)**

### Question 3

Let  $A$  have columns

$$v_1 = (1, 1, 0)^T, \quad v_2 = (1, 0, 1)^T, \quad v_3 = (0, 1, 1)^T.$$

- (a) Apply Gram-Schmidt to obtain an orthonormal basis and hence a QR factorisation  $A = QR$  with positive diagonal entries in  $R$ . **(8 marks)**
- (b) Prove that every matrix in  $O_2(\mathbb{R})$  is either a rotation or a reflection. **(5 marks)**
- (c) Prove that  $O_n(\mathbb{Z})$  consists exactly of the signed permutation matrices. Deduce that it is a subgroup of  $O_n(\mathbb{R})$  and that

$$|O_n(\mathbb{Z})| = 2^n n!.$$

**(7 marks)**

## Question 4

- (a) Let  $H = \{g^2 : g \in G\}$ . Prove that  $H$  is a subgroup when  $G$  is abelian, and prove that  $H = G$  when  $G$  is finite of odd order. **(6 marks)**
- (b) Let  $p, q$  be distinct primes. Prove that  $z \mapsto z^q$  is an automorphism of the group of  $p$ th roots of unity. **(4 marks)**
- (c) Determine the possible cycle shapes of elements of order 3 in  $S_8$  and count all such elements. **(4 marks)**
- (d) Prove that the number of elements of order  $p$  in a finite group is divisible by  $p - 1$ . **(3 marks)**
- (e) Prove Cayley's theorem for finite groups. **(3 marks)**

# **MATH40003 - Linear Algebra and Group Theory**

## **REVISED HISTORICAL COVERAGE MOCK 6**

Seven-Year Mixed Transfer Pressure - unfamiliar wording, familiar course methods

Time allowed: 2 hours

Total marks: 80

Questions: 4 questions, each worth 20 marks

Conditions: Closed book. Start each question on a new sheet. Credit is given only for justified steps.

**Instructions.** You may use results from earlier parts of a question in later parts without

reproving them. Let  $F$  be a field unless another field is stated. The paper has been rebuilt from the 2020–2026 official examination ability range; its overall workload is calibrated to the 2026 two-hour paper.

*Timed student version - audit notes are kept in the separate answer and calibration files.*

## Question 1

Let  $X = \{1, 2, 3, 4, 5, 6\}$ , let  $V$  be the vector space of functions  $X \rightarrow F$ , and let  $\tau = (1\ 2)(3\ 4\ 5)$ , fixing 6. Define  $T(f) = f \circ \tau$ .

- (a) Prove  $V \cong F^6$  and give the delta-function basis. **(4 marks)**
- (b) Prove that  $T$  is linear and find its matrix in the delta-function basis. **(4 marks)**
- (c) Find a basis for  $\ker(I - T)$ . **(4 marks)**
- (d) Under the coordinate isomorphism, prove

$$\operatorname{im}(I - T) = \{x \in F^6 : x_1 + x_2 = 0, x_3 + x_4 + x_5 = 0, x_6 = 0\},$$

and find its rank. **(4 marks)**

- (e) Determine  $\ker(I - T) \cap \operatorname{im}(I - T)$  when  $\operatorname{char} F$  is neither 2 nor 3, when it is 2, and when it is 3. **(4 marks)**

## Question 2

Let  $A : V \rightarrow V$  be linear and satisfy  $A^3 = A$ .

(a) Prove that  $A^2$  is idempotent,  $\ker A^2 = \ker A$ , and  $\operatorname{im} A^2 = \operatorname{im} A$ . **(4 marks)**

(b) Assume 2 is invertible and define

$$P_0 = I - A^2, \quad P_+ = \frac{A^2 + A}{2}, \quad P_- = \frac{A^2 - A}{2}.$$

Prove that the  $P$ 's are pairwise orthogonal idempotents whose sum is  $I$ , and identify their images. **(10 marks)**

(c) Deduce that

$$V = \ker A \oplus \ker(A - I) \oplus \ker(A + I)$$

and that  $A$  is diagonalisable. **(3 marks)**

(d) Over a field of characteristic 2, give a non-diagonalisable  $2 \times 2$  matrix satisfying  $A^3 = A$ , and verify both statements. **(3 marks)**

### Question 3

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

- (a) Find an orthogonal diagonalisation of  $A$ . **(6 marks)**
- (b) Find  $B$  such that  $A = BB^T$ . **(3 marks)**
- (c) Let  $S$  be any real symmetric matrix with eigenvalues  $\lambda_i$ . Prove the maximum and minimum norm formulae on the unit sphere. **(5 marks)**
- (d) Prove that every invertible real matrix has a QR factorisation with an orthogonal first factor and an upper-triangular second factor with positive diagonal. **(6 marks)**

**Question 4**

(a) In  $S_6$ , simplify

$$(1\ 2\ 3\ 4\ 5\ 6)(1\ 2\ 3\ 4\ 5)(1\ 2\ 3\ 4)(1\ 2\ 3)$$

into disjoint cycles, and determine its order and sign.

**(4 marks)**

(b) What is the maximum order of an element of  $S_9$ ? Justify your answer.

**(4 marks)**

(c) Determine whether each group is cyclic, with proof:

$$S_5, \quad (\mathbb{Z} \times \mathbb{Z}, +), \quad \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}, \quad \text{rotations of a regular } m\text{-gon.}$$

**(5 marks)**

(d) Let every element of a group  $G$  have finite order. If  $H \subseteq G$  is non-empty and closed under multiplication, prove that  $H$  is a subgroup.

**(7 marks)**