

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Linear Algebra and Group Theory

Date: Tuesday, May 6, 2025

Time: Start time 14:00 – End time 17:00 (BST)

Time Allowed: 3 hours

This paper has 6 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

You may use results from previous parts in your solutions to later parts without solving them.

Let F be a field throughout.

The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \dots\}$.

The set of $m \times n$ matrices with entries in F is $M_{m,n}(F) = M_{m \times n}(F)$, and also $M_n(F) = M_{n,n}(F)$. You may use any of these notations in your answers.

If $A \in M_{m,n}(F)$, $rk A$ denotes the rank of A .

1. In parts (a)–(c), consider the following matrix over F :

$$\begin{pmatrix} 0 & 1 & a & -1 \\ b & c & d & 0 \end{pmatrix}$$

- (a) For which values of a, b, c, d is the matrix in row-echelon form? (2 marks)
- (b) For which values of a, b, c, d is it in reduced row-echelon form? (2 marks)
- (c) Give a basis for the row space and the column space of the matrix, depending (as necessary) on the values of a, b, c, d . (6 marks)
- (d) Let A and B be matrices. Prove or give a counterexample to the following statements:
 - (i) $\text{rowspace}(A + B) \subseteq \text{rowspace}(A) + \text{rowspace}(B)$, when A, B are of the same size. (2 marks)
 - (ii) $\text{nullspace}(AB) \supseteq \text{nullspace}(B)$, when AB is well-defined. (2 marks)
 - (iii) $\text{rowspace}(AB) \subseteq \text{rowspace}(B)$, when AB is well-defined. (3 marks)
 - (iv) Suppose $A \in M_{m,n}(F)$ and $B \in M_{n,m}(F)$ for $m \geq n$. Then $AB = 0$ implies $BA = 0$. (3 marks)

(Total: 20 marks)

2. Let V be a vector space over the field F .

(a) Let U_1 and U_2 be two subspaces of V . Give an example to show that the union $U_1 \cup U_2$ does not have to be a subspace of V . (3 marks)

(b) Let U_n , for $n \geq 1$, be an increasing sequence of subspaces of V :

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots \subseteq V.$$

Show that the union $\bigcup_{n \in \mathbb{N}} U_n$ is a subspace of V . (5 marks)

(c) Define what it means for V to be *finite dimensional*. (1 mark)

(d) Show that if V is finite dimensional, then for every chain of inclusions of subspaces

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots \subseteq V$$

there is an $N \in \mathbb{N}$ such that $U_N = U_{N+1} = U_{N+2} = \cdots$ (6 marks)

You may use part (b) and any facts from lectures about finite dimensional vector spaces, if you state them carefully.

(e) Consider the vector space $F[x]$ of polynomials with coefficients in F . Find a sequence of subspaces

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots \subseteq F[x]$$

such that $U_i \neq U_{i+1}$ for all i . (5 marks)

(Total: 20 marks)

3. Consider a square in the plane, with vertices $X = \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$. Let V be the set of all functions $\{f : X \rightarrow F\}$. Equip V with the addition of functions, $(f + g)(v) := f(v) + g(v)$, for $f, g \in V$, and scalar multiplication, $(\lambda f)(v) := \lambda f(v)$, for $f \in V$ and $\lambda \in F$.

We first study V , which only involves the set X , and afterwards we will use the interpretation as the vertices of a square.

- (a) Show that the map $\varphi : V \rightarrow F^4$, $\varphi(f) = \begin{pmatrix} f(0, 1) \\ f(1, 0) \\ f(0, -1) \\ f(-1, 0) \end{pmatrix}$ is a bijection. (3 marks)
- (b) Show that V is a vector space and φ is an isomorphism. Which function in V is the zero element? (6 marks)
[Hint: Using (a), it is possible to explain this without checking the individual axioms.]
- (c) Show that a basis of V is given by the functions $f_{0,1}, f_{1,0}, f_{0,-1}$, and $f_{-1,0}$, where

$$f_{a,b}(v) = \begin{cases} 1, & \text{if } v = (a, b), \\ 0, & \text{otherwise.} \end{cases}$$
 (3 marks)
- (d) Let $R : X \rightarrow X$ be given by a counterclockwise rotation of the square by $\pi/2$. Consider the transformation $T : V \rightarrow V$, $T(f) = f \circ R$. In the basis B from (c), find the matrix of T . (4 marks)
- (e) Find the kernel, image, and rank of T . (4 marks)

(Total: 20 marks)

4. (a) Let $V = \mathbb{R}^3$ (viewed as a vector space over $\mathbb{R}$) and $T : V \rightarrow V$ be the linear transformation defined by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_3 \\ x_1 + 5x_2 + x_3 \\ -x_1 - 6x_2 - 2x_3 \end{pmatrix}.$$

Let $E = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and A is the matrix of T with respect to the basis E .

Note that this was denoted by $[T]_E$, ${}_E[T]_E$, or $[T]_{EE}$ in class.

- (i) Write down A , explaining your answer. (3 marks)
 - (ii) Find the eigenspaces of T (show the details of your calculation), and determine whether T is diagonalisable or not (explain your answer). (8 marks)
- (b) Let $n \in \mathbb{N}$ and let $A \in M_n(F)$. Let $k, \ell \leq n$. A $(k \times \ell)$ -submatrix of A is a $k \times \ell$ matrix obtained from A by choosing k rows, ℓ columns, and removing all the other rows and columns. If $k = \ell$ we say that the submatrix is square of order k .
- (i) Prove that $\text{rk } A < k$ if and only if all its square submatrices of order k have vanishing determinant.
[Hint: to solve this, you may use without proof that the rank of a matrix is the size of a maximal set of linearly independent rows and that this is equal to the size of a maximal set of linearly independent columns.] (5 marks)
 - (ii) Let M be a square submatrix of A of order $k < n$ and such that $\det(M) \neq 0$. Prove that $\text{rk } A = k$ if and only if all the square submatrices of A of order $k + 1$ containing M (as a submatrix) have vanishing determinant.
[Hint: if $\text{rk } A > k$, show that there must be $k + 1$ linearly independent rows containing M . Similarly show that in the submatrix formed by these rows, there must be $k + 1$ linearly independent columns containing M .] (4 marks)

(Total: 20 marks)

5. (a) Let $A \in M_3(\mathbb{R})$ be defined as follows

$$\begin{pmatrix} 0 & 0 & 4 \\ 1 & 4 & -1 \\ 0 & 3 & 4 \end{pmatrix}.$$

Let v_1 , v_2 , and v_3 be the first, second, and third columns of A respectively.

- (i) Use the Gram-Schmidt process on $B = \{v_1, v_2, v_3\}$ to find an orthonormal basis of $\mathbb{R}^3$.
(For this part, you may assume without proof that B forms a basis of $\mathbb{R}^3$.) (4 marks)
 - (ii) Let O be the orthonormal basis you found in the previous part. Express each vector of B as a linear combination of the vectors in O . (3 marks)
 - (iii) Find an orthogonal matrix Q and an upper-triangular matrix $R = (r_{ij})$ with $r_{ii} > 0$ for all $i = 1, 2, 3$ such that $A = QR$. (3 marks)
- (b) Let $n \in \mathbb{N}$ and let $O_n(\mathbb{Z})$ be the set consisting of all the $n \times n$ orthogonal matrices with entries in $\mathbb{Z}$.
- (i) Prove that $O_n(\mathbb{Z})$ is a subgroup of the group of all $n \times n$ orthogonal matrices over $\mathbb{R}$ (where the binary operation is the usual matrix multiplication). (3 marks)
 - (ii) We define a *permutation matrix* to be a square matrix that has exactly one entry equal to 1 in each row and each column with all other entries equal to 0.
Prove that $O_n(\mathbb{Z})$ contains all the $n \times n$ permutation matrices. (3 marks)
 - (iii) Prove that $O_n(\mathbb{Z})$ is a finite group and compute its order.
[Hint: If you wish, you may assume without proof that the $n \times n$ permutation matrices form a group that is isomorphic to the symmetric group S_n .] (4 marks)

(Total: 20 marks)

6. (a) (i) Write the permutation $(12345)(1234)(123)(12)$ as a product of *disjoint* cycles. Hence, or otherwise, determine the order of this permutation. (2 marks)
- (ii) What is the maximum order of an element of the symmetric group S_9 ? Justify your answer. (4 marks)
- (b) For each of the following groups determine whether it is cyclic. Justify your answers.
- (i) The symmetric group S_5 . (3 marks)
- (ii) The group of matrices
- $$\left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$
- where the group operation is multiplication of matrices. (3 marks)
- (iii) The group of rotations of the regular m -gon. (1 mark)
- (c) Let G be a group such that every element of G has finite order. Let $H \subseteq G$ be a non-empty subset such that for any $x, y \in H$ we have $xy \in H$. Is H necessarily a subgroup of G ? Give a proof or a counterexample. (7 marks)

(Total: 20 marks)

Solutions

1.

seen ↓

(a) We must have $b = c = 0$, and $d = 1$ or $d = 0$, to be row-echelon. There is no condition on a .

2, A

(b) In addition to the conditions in part (a), when $d = 1$, we need also $a = 0$. (Note: if the student missed the possibility $d = 0$ in part (a), they should not be penalised again here.)

seen ↓

(c) Note that the matrix has rank at least one because the first row is nonzero. The rank is exactly one if and only if the second row is a multiple of the first row. For this to be the case, we need the multiple to be zero since the last entry, 0, is only the zero multiple of -1 . So the rank is exactly one if and only if $b = c = d = 0$.

2, A

seen ↓

In the rank one case, then, a basis for the row space is just the first, nonzero row.

2, B

In this case, the first row is nonzero (but not the second), so a basis for the column space is the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

1, A

Finally, in the rank two case, so that some of b, c, d are nonzero, then the two rows form a basis for the row space, as they are linearly independent.

1, A

The column space is all of $\mathbb{R}^2$ so a basis is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

1, C

1, C

(d)

seen ↓

(i) Each row of $A + B$ is the sum of a row of A and a row of B , so in the sum of the row spaces of A and B . As the rows of $A + B$ span the row space (by definition), the inclusion follows.

2, B

(ii) The null space consists of vectors v such that $Bv = 0$.

1, A

Now if $B = 0$ then $AB = 0$ too.

1, A

(iii) Write $A = (a_{ij})$, and let the rows of B be denoted $B_1, \dots, B_m$. Then the i -th row of AB , by the definition of matrix multiplication, is $a_{i1}B_1 + \dots + a_{im}B_m$. This is in the row space.

3, B

(Alternative solution: The row space is the perpendicular space to the null space, i.e., the vectors with zero dot product with the null space. So since $\text{nullspace}(AB) \supseteq \text{nullspace}(B)$, we get the opposite inclusion on row spaces.)

(iv) This is false. A counterexample is given by $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

3, C

(Total: 20 marks)

2.

seen ↓

(a) You could take $V = \mathbb{R}^2$ and $U_1 = \text{span}\{e_1\}$ and $U_2 = \text{span}\{e_2\}$ (the x - and y -axes). Then $e_1, e_2 \in U_1 \cup U_2$ but $e_1 + e_2 \notin U_1 \cup U_2$, so the union is not a subspace.

3, A

(b) • First note that $0_V \in \bigcup_n U_n$ since $0_V \in U_1$.

unseen ↓

• Next, if $u \in \bigcup_n U_n$ and $\lambda \in \mathbb{R}$ then, by definition of the union, $u \in U_n$ for some n . Since U_n is a subspace $\lambda u \in U_n$, and therefore $\lambda u \in U_n$.

• Finally, if $u_1, u_2 \in U$, then $u_1 \in U_{n_1}$ and $u_2 \in U_{n_2}$ for some n_1, n_2 . If $n = \max\{n_1, n_2\}$, then since the sequence is increasing, $U_{n_1}, U_{n_2} \subseteq U_n$, so $u_1, u_2 \in U_n$. Since U_n is a subspace $u_1 + u_2 \in U_n$, and so $u_1 + u_2 \in U$.

5, B

(c) V is finite dimensional if it is the span of a finite set of vectors, or equivalently, if it admits a finite basis (either answer is fine).

seen ↓

(d) By part (b) the union $U = \bigcup_n U_n$ is a subspace of V . It is a fact from lectures that a subspace of a finite dimensional vector space is itself finite dimensional. Therefore U is finite dimensional, and there are vectors $u_1, \dots, u_k \in U$ such that $U = \text{span}\{u_1, \dots, u_k\}$. By definition of the union each u_i is in U_{n_i} for some n_i . Let $N = \max\{n_1, \dots, n_k\}$. Then for any $M \geq N$, we have $U_N \subseteq U_M \subseteq U = \text{span}\{u_1, \dots, u_k\} \subseteq U_N$. So $U_N = U_{N+1} = U_{N+2} = \dots$.

1, A

unseen ↓

An alternative proof is to more heavily use some dimension theory from the lectures: Say that $\dim(V) = n$. Since each $U_i \subseteq V$ is a subspace, it is a fact from lectures that U_i must also be finite dimensional with $\dim(U_i) \leq n$. By the same fact from lectures, $\dim(U_i) \leq \dim(U_{i+1})$. This means we have inequalities

$$\dim(U_1) \leq \dim(U_2) \leq \dots \leq \dim(U_i) \leq \dots \leq n.$$

From this we can see that there must be an N such that $\dim(U_N) = \dim(U_{N+1}) = \dots$

Now, it is also a fact from lectures that if U is a subspace of U' , and $\dim(U) = \dim(U')$ then $U = U'$. From this it follows that $U_N = U_{N+1} = \dots$

6, D

(e) One natural example is

unseen ↓

$$U_n = \mathbb{R}[x]_{\leq n} = \{\text{polynomials of degree } \leq n\}.$$

These satisfy $\mathbb{R}[x]_{\leq 1} \subseteq \mathbb{R}[x]_{\leq 2} \subseteq \dots$ and $\mathbb{R}[x]_{\leq n} \neq \mathbb{R}[x]_{\leq n+1}$ for all n .

5, B

(Total: 20 marks)

3.

unseen ↓

- (a) This is just saying that a function is uniquely determined by its values on the domain. In more detail, the inverse to this map is the map sending $(a, b, c, d) \in \mathbb{R}^4$ to the unique function f such that $f(0, 1) = a, f(1, 0) = b, f(0, -1) = c$, and $f(-1, 0) = d$. It is clear that the composition in either order of these two maps is the identity.

3, A

unseen ↓

- (b) Since $\varphi : V \rightarrow \mathbb{R}^4$ is a bijection, provided we have $\varphi(f + g) = \varphi(f) + \varphi(g)$ and $\varphi(\lambda f) = \lambda\varphi(f)$, then the axioms for V to be a vector space will each of them follow from the axioms of $\mathbb{R}^4$, for 0 the element $\varphi^{-1}(0)$. Moreover, by definition φ is then an isomorphism. So we check this, writing for convenience the elements of $\mathbb{R}^4$ as transposed tuples $(a, b, c, d)^t$:

$$\varphi(f + g) = (f(0, 1) + g(0, 1), f(1, 0) + g(1, 0), f(0, -1) + g(0, -1), f(-1, 0) + g(-1, 0))^t.$$

This clearly equals

$$\varphi(f) + \varphi(g) = (f(0, 1), f(1, 0), f(0, -1), f(-1, 0)) + (g(0, 1), g(1, 0), g(0, -1), g(-1, 0))^t.$$

Similarly,

$$\varphi(\lambda f) = (\lambda f(0, 1), \lambda f(1, 0), \lambda f(0, -1), \lambda f(-1, 0))^t = \lambda\varphi(f).$$

The zero element is the zero function, since φ sends it to $(0, 0, 0, 0)^t$.

6, D

seen ↓

- (c) Since φ is an isomorphism, a basis for V is obtained as φ^{-1} of a basis of $\mathbb{R}^4$. Using the standard basis of $\mathbb{R}^4$, we get the basis of V given by the functions $f_{0,1}, f_{1,0}, f_{0,-1}$, and $f_{-1,0}$.

3, D

seen ↓

- (d) The basis is permuted by this rotation and we get the following permutation matrix, using the given ordering of the basis:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In more detail, $f_{a,b} \circ R = f_{a',b'}$ where $(a', b') = R^{-1}(a, b)$ (note the inverse!). Then $T(f_{0,1}) = f_{1,0}$, $T(f_{1,0}) = f_{0,-1}$, $T(f_{0,-1}) = f_{-1,0}$, and $T(f_{-1,0}) = f_{0,1}$.

4, B

- (e) The transformation T is invertible, with inverse given by the clockwise rotation by $\pi/2$. Accordingly the kernel is zero, the image is all of V , and the rank is the dimension of V , which is four.

sim. seen ↓

(Total: 20 marks)

4, A

4. (a) (i) The images of the vectors in E are

seen ↓

$$T(e_1) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, T(e_2) = \begin{pmatrix} 0 \\ 5 \\ -6 \end{pmatrix}, T(e_3) = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

Thus

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 5 & 1 \\ -1 & -6 & -2 \end{pmatrix}.$$

3, A

- (ii) The characteristic polynomial of A is $\det(XI_3 - A) = (X - 3)(X - 2)(X + 1)$. Thus there are three eigenvalues: $-1, 2, 3$.

3, A

The eigenspace for -1 is

$$E_{-1} = \ker(-I_3 - A) = \ker \begin{pmatrix} -2 & 0 & -2 \\ -1 & -6 & -1 \\ 1 & 6 & 1 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

The eigenspace for 2 is

$$E_2 = \ker \begin{pmatrix} 1 & 0 & -2 \\ -1 & -3 & -1 \\ 1 & 6 & 4 \end{pmatrix} = \text{Span} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

The eigenspace for 3 is

$$E_3 = \ker \begin{pmatrix} 2 & 0 & -2 \\ -1 & -2 & -1 \\ 1 & 6 & 5 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

3, B

There is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A . Hence A is diagonalisable.

2, A

- (b) *Note to the markers: give (at least) partial credit for solutions that are written in less detail than the ones below. The main idea behind both proofs below is that a matrix has non-zero determinant if and only if its rows (and columns) are linearly independent*

unseen ↓

- (i) We prove that there is a square submatrix of order k with non-zero determinant if and only if the rank of A is at least k . Suppose there is a square submatrix M of order k with non-zero determinant. Let M' be the $(n \times k)$ -submatrix consisting of the columns chosen for M . Then the rank of M' is k because the k rows chosen for M are linearly independent as M has rank k and these form a maximal set of linearly independent rows for M' . Therefore, there are at least k linearly independent columns in A and so its rank is at least k .

3, B

Conversely, suppose the rank is at least k , then A has (at least) k linearly independent columns. Let M' be an $(n \times k)$ -submatrix of A consisting of a set of k linearly independent columns. This matrix has rank k so it has k linearly independent rows. If these are chosen and the rest is deleted, we get a square submatrix of order k with non-vanishing determinant.

2, C

- (ii) The submatrices in this part of the question are a subset of those in the previous part, so we need only prove that if their determinant vanishes, $\text{rk} A \leq k$.

unseen ↓

We prove the contrapositive, that is, if the rank of A is at least $k + 1$, then one of the submatrices in the question is non-singular. Following the hint, the rows of A containing M as a submatrix are linearly independent but cannot span the entire row space of A , so there is another row that is not in the span of these. Considering these $k + 1$ rows alone, gives a submatrix, of A of rank $k + 1$.

2, D

Let M' be the submatrix of rank $k + 1$ formed by the rows of M and the additional row (as explained above). Similarly to what done before (but this time using columns), the column space of M' has dimension $k + 1$. The columns containing M (in M') span a space of dimension k , so there must be another column in M' that is not in the span of those containing M . Adjoining this column to those containing M gives the sought non-singular submatrix of order $k + 1$ containing M .

2, D

(Total: 20 marks)

5. (a) (i) By the Gram-Schmidt process we have

seen ↓

$$\begin{aligned} w_1 &= v_1, \\ w_2 &= v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} w_1 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \\ w_3 &= v_3 - \frac{v_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_3 \cdot w_2}{\|w_2\|^2} w_2 = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

Normalising the vectors above, we find the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

4, A

(ii) Calling the vectors of $\mathcal{B}$ u_1 , u_2 , and u_3 respectively, we get that

$$v_1 = u_1, \quad v_2 = 4u_1 + 3u_2, \quad v_3 = -u_1 + 4u_2 + 4u_3$$

3, A

(iii) By the above, it is immediate to see that

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}.$$

3, B

(b) (i) We use the subgroup test. Let $H = O_n(\mathbb{Z})$. Clearly $H \neq \emptyset$ as it contains the identity matrix. If two matrices with integer entries are multiplied, the result is another matrix with integer entries, so H is closed under multiplication. It is also closed under taking inverses because, for the matrices in $O_n(\mathbb{R})$, the inverse is just the transpose and if $A \in M_n(\mathbb{Z})$, then $A^T \in M_n(\mathbb{Z})$. Thus, if $A \in H = O_n(\mathbb{R}) \cap M_n(\mathbb{R})$, $A^{-1} = A^T \in H$.

seen ↓

3, A

(ii) A permutation matrix has integer entries and its columns are pair-wise orthonormal because they are just a rearrangement of the standard basis. Hence $O_n(\mathbb{Z})$ contains all the permutation matrices.

unseen ↓

3, C

(iii) Let $a_1, \dots, a_n$ be the columns of $A \in O_n(\mathbb{Z})$. Then for all $i, j \in \{1, \dots, n\}$

unseen ↓

$$a_i \cdot a_j = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Moreover for all $i \in \{1, \dots, n\}$ we have $a_i \cdot a_i = \sum_{k=1}^n a_{ik}^2 = 1$. This means that for each column there only is one non-zero entry and its value is either 1 or -1 . For $i, j \in \{1, \dots, n\}$ with $i \neq j$, $0 = \sum_{k=1}^n a_{ik} a_{jk}$, this implies that the non-zero entry of a_i is not in the same row as the non-zero entry of a_j . We conclude that $O_n(\mathbb{Z})$ consists of all those matrices that have exactly one entry equal to 1 or -1 in each row and each column with all other entries equal to 0.

Suppose we are choosing a matrix in $O_n(\mathbb{Z})$. We have $2n$ choices for the first column. After this, we have $2(n-1)$ choices for the second column because we are allowed to put ± 1 in any of the $n-1$ rows where the first column is 0. Similarly there are $2(n-2)$ choices for the third column. This way there are exactly $2(n-i+1)$ choices for the i th column and so the order of the group is $2^n \cdot n!$.

4, D

Alternatively one can argue that if H is the subgroup of permutation matrices (which has order $n!$), the left cosets of H are exactly

$$\text{diag}(a_1, \dots, a_n)H$$

where $a_i \in \{-1, 1\}$. So there are exactly as many left cosets of H as there are choices for $\text{diag}(a_1, \dots, a_n)$, that is 2^n .

(Total: 20 marks)

6. (a) (i) This permutation is $(15)(24)$, hence its order is 2.
- (ii) Every element of S_9 can be written as a product of cycles of lengths $n_1, \dots, n_r$ such that $n_1 + \dots + n_r = 9$, where $n_i \geq 1$ for all i . The order of any permutation of this cycle type is the least common multiple of $n_1, \dots, n_r$. The maximum of l.c.m. is 20 which is attained for $r = 2$, $n_1 = 4$, $n_2 = 5$.
- (b) (i) Possible cycle types of elements of S_5 are 2, 3, 4, 5, (2,2), (2,3), where we omit cycles of length 1. The orders of elements of these cycle types are 2, 3, 4, 5, 2, 6, respectively. In particular, S_5 does not contain elements of order $|S_5| = 120$. An alternative way to see that S_5 is not cyclic is to note that S_5 is not commutative. Indeed, $(12)(13) \neq (13)(12)$.
- (ii) Write $M_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Then we have $M_n M_k = M_{n+k}$ for all $n, k \in \mathbb{Z}$. This implies $(M_n)^{-1} = M_{-n}$. In particular, our group is $\{(M_1)^n | n \in \mathbb{Z}\}$, therefore it is an infinite cyclic group.
- (iii) Let ρ be the rotation through the angle $2\pi/m$. It preserves the regular m -gon. Moreover, every rotation preserving the regular m -gon is ρ^i , where $i = 0, 1, \dots, m-1$. Thus these rotations form a cyclic group of order m with generator ρ .
- (c) H is necessarily a subgroup of G . Take any $h \in H$ and consider the sequence $h, h^2, h^3, \dots$. By assumption, this sequence is contained in H . Moreover, h has finite order, say n . Then $h^n = e$, so that $e \in H$. Next, we have $h^{-1} = h^{n-1}$, so $h^{-1} \in H$. Thus H contains e and is closed under taking inverses, and so is a subgroup of G .

sim. seen ↓

2, A

unseen ↓

4, B

unseen ↓

3, A

unseen ↓

3, A

seen ↓

1, A

unseen ↓

7, C

(Total: 20 marks)

Total A marks: 46 of 48 marks

Total B marks: 34 of 30 marks

Total C marks: 17 of 18 marks

Total D marks: 23 of 24 marks

MATH40003 Linear Algebra and Group Theory Markers comments

Question 1 This question was a conceptual question. For part a) of the question it was a common mistake that some students did not consider the case $d=0$. In part c) of the question the common mistake was to not consider the case when the rank of the matrix would be one. Also some students did not write the complete argument for why the two rows would be linearly independent if any of b, c, d is non zero. For the first two sub-part of question d, majority of students performed well. for the last sub-part of this question some students concluded the wrong answer as they assumed A is a invertible square matrix. Also some students did not take the hypothesis $m \geq n$ while constructing the counter example.

Question 2 This was a very proof-based question, but dealing with basic concepts, mainly the subspace axioms and dimension. Students seemed to understand these concepts and did relatively well on the question. In part (a) one common mistake was to argue abstractly without giving an example, as asked for. Some students also mixed-up subspaces and subgroups and wrote down examples involving closure under multiplication. In part (b) those students who tried to directly verify the subspace axioms did very well. But many students quite reasonably thought to use induction, showing that the union of the first n subspaces is a subspace for all n ; unfortunately, this says nothing about the infinite union, so this approach would not work (although the noticing this is subtle). Almost all students could give a reasonable definition of finite dimensional, answering part (c). Part (d) was the hardest part of the question - almost all students tried a proof that that started by noticing that the sequence of subspaces had (weakly) increasing dimension. A lot of students incorrectly stated that the union of all the subspaces would be the whole space V , and some incorrectly negated the given statement (towards a contradiction), saying that it meant the chain of inclusions of subspaces was strict at every index. There was a surprising amount of discussion of limits and convergence (even some epsilons) which was not necessary but was not penalised (except when students talked about limits/convergence of subspaces, which is not itself meaningful). Part (e) was answered very well, with most students giving the same answer (polynomials up to a given degree). The issues with this question were often notational, and a lot of students had the right idea but wrote down things that didn't quite make sense.

Question 3 This was one of the more abstract, conceptual questions, which did not require much computation but allowed students to demonstrate a greater understanding of the concepts of bases, linear isomorphism, and linear transformations and their relationship with matrices. In spite of this many students did quite well. Common points of confusion included: In Q3a, the map is not yet known to be linear so injectivity has to be shown directly rather than by computing the kernel. In Q3b, the set V is not presented as a subset of any other vector space, so it cannot be shown to be a vector space by demonstrating closure under the operations. Instead, either the axioms need to be shown, or the axioms can be deduced to hold from those of F^4 via the bijection from Q3a, once we know that it also preserves addition and scalar multiplication. Also, many students wanted to show the map was a group homomorphism rather than a linear map, so linearity was missed and multiplicative instead of the appropriate additive notation was often used. Some students noted that being bijective to a vector space implies a set is a vector space, but this argument is also incomplete since there is no guarantee the resulting vector space structure is the same as the given one. In Q3c, there were no major difficulties, but many students did not make use of the isomorphism from Q3b, instead showing directly the set is a basis, which takes more work. In Q3d, many students produced an incorrect matrix such as the inverse; there was some confusion on the difference between the action of rotation on the set of vertices and on the basis of functions on the vertices. In Q3e, there was some confusion about writing the kernel and the image as in V rather than the vector space F^4 of tuples.

Question 4 This question was divided into two parts, (a) and (b), each assessing different levels of understanding. Part (a) focused on concrete computational skills related to linear transformations, allowing students to demonstrate their ability to apply core concepts in a straightforward way. In contrast, part (b) required more abstract reasoning and the ability to construct a rigorous proof involving fundamental properties of matrices. Part (a) was generally very well answered. The majority of students who attempted it obtained full or nearly full marks. The most common mistakes were minor and typically involved arithmetic or algebraic errors, rather than misunderstandings of the underlying concepts. Part (b) proved to be significantly more challenging. Only a small number of students attempted it, and even fewer produced a complete and correct solution. A frequent point of confusion was the distinction between the linear independence of the rows of a submatrix and that of the rows of the original matrix. Moreover, the fact that part (b.1) implies

one direction of the result required in part (b.2) was largely overlooked. In several cases, students also appeared uncertain about the precise statement they were meant to prove.

- Question 5 Part a was generally solved well, although a few people could have spent more words justifying (a)(iii) Part b(i) Essentially, it was required to show that $I_n \in O_n(\mathbb{Z})$ and that AB^{-1} have integer entries whenever A, B are two orthonormal matrices with integer entries . Most people missed this point and showed AB^{-1} is orthonormal (but this is trivial because $O_n(\mathbb{R})$ is a group) (ii) Generally solved well, sometimes more justification was needed to show permutation matrices are orthogonal (iii) only a few recognised that the matrices in $O_n(\mathbb{Z})$ could have also -1 as an entry.
- Question 6 This question was done well by the majority of students. One difficulty was in identifying a group with the infinite cyclic group: in this case one allows all *integer* powers of the generator and not just positive integer powers. "Rotations of a regular m -gon" means "rotations preserving the regular m -gon". This confused some students, although this part was worth only 1 mark.