

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2026

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Linear Algebra and Group Theory

Date: Wednesday, 29 April, 2026

Time: Start time 14:00 – End time 16:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

You may use results from previous parts in your solutions to later parts without solving them.

Let F be a field throughout.

The set of $m \times n$ matrices with entries in F is $M_{m,n}(F) = M_{m \times n}(F)$, and also $M_n(F) = M_{n,n}(F)$. You may use any of these notations in your answers.

If $A \in M_{m,n}(F)$, then $rk(A)$ denotes the rank of A .

1. Let $D : F^5 \rightarrow F^4$ be the linear transformation defined by

$$D \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} a + e - d \\ b - a \\ c - b - e \\ d - c \end{pmatrix}.$$

Let $G : F^4 \rightarrow F^5$ be the linear transformation defined by

$$G \begin{pmatrix} s \\ t \\ u \\ r \end{pmatrix} = \begin{pmatrix} s - t \\ t - u \\ u - r \\ r - s \\ s - u \end{pmatrix}.$$

- (a) Write down the matrix of D with respect to the standard bases of F^5 and F^4 . (3 marks)
- (b) Find a basis for $\ker(D)$ and determine $\dim \ker(D)$. (6 marks)
- (c) Write down the matrix of G with respect to the standard bases of F^4 and F^5 . Find $\ker(G)$ and determine $\dim(\operatorname{im}(G))$. (3 marks)
- (d) Assume that 2 is invertible in F . Prove that $\ker(D) \cap \operatorname{im}(G) = \{0\}$.
[Hint: You may use that $\ker(D) = \{(a, a, a + e, a + e, e)^t : a, e \in F\}$, in case you did not find it in (b). Furthermore, notice that the sum of the first four entries of $G((s, t, u, r)^t)$ is 0, and express the last entry in terms of other entries.] (4 marks)
- (e) Assume that 2 is invertible in F . Deduce that

$$F^5 = \ker(D) + \operatorname{im}(G),$$

and determine $rk(D)$. (2 marks)

- (f) Now assume that $2 = 0$ in F . Exhibit a nonzero element of $\ker(D) \cap \operatorname{im}(G)$. (2 marks)

(Total: 20 marks)

2. Let V be the vector space

$$V = F[x]_{\leq 3} = \{p(x) \in F[x] \mid \deg p(x) \leq 3\},$$

with ordered basis $B = (1, x, x^2, x^3)$.

Define $J : V \rightarrow V$ by $(Jp)(x) = p(-x)$.

- (a) Show that J is linear and write down its matrix in the basis B . (3 marks)
- (b) Define

$$V^+ = \{p(x) \in V \mid p(-x) = p(x)\}, \quad V^- = \{p(x) \in V \mid p(-x) = -p(x)\}.$$

Show that V^+ and V^- are subspaces of V , and find bases for them (your answer may depend on whether or not $2 = 0$ in F).

Determine $\dim(V^+ \cap V^-)$, and state when $V^+ \cap V^- = 0$ (depending on F). (7 marks)

Now define $D : V \rightarrow V$ by $D(p)(x) = p'(x)$ (the formal derivative).

- (c) Write down the matrix of D in the basis B . (2 marks)
- (d) Prove that $D \circ J = -J \circ D$. Deduce that $D(V^+) \subseteq V^-$ and $D(V^-) \subseteq V^+$. (4 marks)
- (e) Determine $\text{rk}(D|_{V^+})$ and $\text{rk}(D|_{V^-})$. Your answer should depend on whether or not $2 \neq 0$ and whether or not $3 \neq 0$ in F . (4 marks)

(Total: 20 marks)

3. (a) Let $A \in M_3(\mathbb{R})$ be given by

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

(i) Find the eigenvalues of A . (3 marks)

(ii) Find an orthogonal matrix $Q \in M_3(\mathbb{R})$ and a diagonal matrix $D \in M_3(\mathbb{R})$ such that $A = QDQ^{-1}$. (4 marks)

(iii) Find a matrix $B \in M_3(\mathbb{R})$ such that $A = BB^T$. (3 marks)

(b) Let $n \geq 1$. We say that $Z \in M_n(\mathbb{C})$ is *unitary* if $\overline{Z^T}Z = I_n$. Here $\overline{Z^T}$ means the complex conjugate of the transpose of Z . In other words, if $Z = (z_{i,j})$, then the (i, j) -entry of $\overline{Z^T}$ is $\overline{z_{j,i}}$.

Given four matrices $A, B, C, D \in M_n(\mathbb{R})$, we may define a matrix $E \in M_{2n}(\mathbb{R})$ by formula

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We say that a matrix $M \in M_{2n}(\mathbb{R})$ has property (P) if $M = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ for some $A, B \in M_n(\mathbb{R})$ such that the matrix $Z = A + iB \in M_n(\mathbb{C})$ is unitary.

Let $M \in M_{2n}(\mathbb{R})$ be a matrix with property (P).

(i) Prove that M is orthogonal. (3 marks)

(ii) Let I_n denote the identity matrix in $M_n(\mathbb{R})$ and let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{R})$. Prove that $JM = MJ$. (3 marks)

(iii) Let J be as in (b) part (ii). Assume that $Q \in M_{2n}(\mathbb{R})$ is orthogonal and satisfies $JQ = QJ$. Prove that Q has property (P). (4 marks)

(Total: 20 marks)

4. (a) Let σ be the following permutation in S_{11} .

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 4 & 5 & 6 & 7 & 2 & 1 & 10 & 11 & 8 & 9 \end{pmatrix}.$$

- (i) Find a decomposition of σ into a product of disjoint cycles. (4 marks)
 - (ii) Hence or otherwise, find a decomposition of σ into a product of 2-cycles. (2 marks)
 - (iii) Hence or otherwise, compute the sign (or signature) of σ . (2 marks)
- (b) (i) Let G be a group and let $x, y \in G$. We say that x and y are *conjugate* if there is $g \in G$ such that

$$gxg^{-1} = y.$$

Prove that if we set $x \sim y$ if and only if x is conjugate to y , then $\sim$ is an equivalence relation. (3 marks)

- (ii) Prove that, in the notation above, for all $g, x \in G$ and for all $i \geq 1$, $gx^i g^{-1} = (gxg^{-1})^i$. (2 marks)

- (c) Let $n \geq 1$ (and let $G = S_n$). A partition of n is a sequence of positive integers $[m_1, \dots, m_i]$ such that $n = m_1 + \dots + m_i$ and $m_1 \geq \dots \geq m_i$.

If $\sigma \in S_n$, its decomposition into disjoint cycles (including cycles of length one) has an attached partition of n , encoding the length of the cycles in the decomposition.

Note that we insist the sequences of numbers in a partition of n are non-increasing, so if $\sigma \in S_n$, its associated partition is unique and we can denote it by $c(\sigma)$. **For example**, if

$$\sigma = (124)(35) = (124)(35)(6) \in S_6 \text{ we have } c(\sigma) = [3, 2, 1].$$

Prove that $\sigma, \rho \in S_n$ are conjugate in S_n if and only if $c(\sigma) = c(\rho)$. (7 marks)

Hint: prove that if $\sigma = (a_1 a_2 \dots a_\ell) \in S_n$ and $\tau \in S_n$ then $\tau \sigma \tau^{-1} = (\tau(a_1) \tau(a_2) \dots \tau(a_\ell))$.

(Total: 20 marks)

Solutions

1.

seen ↓

(a) From $D(a, b, c, d, e)^t = (a + e - d, b - a, c - b - e, d - c)^t$, we get

$$[D] = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

3, A

(b) Solve $D(a, b, c, d, e)^t = 0$:

seen ↓

$$\begin{aligned} b - a = 0 &\Rightarrow b = a, & d - c = 0 &\Rightarrow d = c, \\ a + e - d = 0 &\Rightarrow c = d = a + e. \end{aligned}$$

Then the third equation is automatic: $c - b - e = (a + e) - a - e = 0$. Hence

$$\ker(D) = \{(a, a, a + e, a + e, e)^t : a, e \in F\} = \text{span}\{(1, 1, 1, 1, 0)^t, (0, 0, 1, 1, 1)^t\},$$

so $\dim \ker(D) = 2$.

6, B

(c) From

seen ↓

$$G(s, t, u, r)^t = (s - t, t - u, u - r, r - s, s - u)^t$$

we obtain

$$[G] = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Now $G(s, t, u, r)^t = 0$ implies $s = t = u = r$, so $\ker(G) = \text{span}\{(1, 1, 1, 1)^t\}$ and, by the rank-nullity theorem, $\dim(\text{im}(G)) = 4 - 1 = 3$.

3, A

(d) Let $v = (a, b, c, d, e)^t \in \text{im}(G)$. Then $v = G(s, t, u, r)^t$ for some $s, t, u, r \in F$, so

unseen ↓

$$a + b + c + d = (s - t) + (t - u) + (u - r) + (r - s) = 0, \quad e = s - u = (s - t) + (t - u) = a + b.$$

Now suppose $v \in \ker(D)$ as well. Using the description of $\ker(D)$ from part (b), we may write

$$v = (a, a, a + e, a + e, e)^t.$$

The condition $a + b + c + d = 0$ becomes $4a + 2e = 0$, and the condition $e = a + b$ becomes $e = 2a$. Substituting gives $8a = 0$. If 2 is invertible in F (as assumed in the question), then $a = 0$, hence $e = 0$ and therefore $v = 0$. So $\ker(D) \cap \text{im}(G) = \{0\}$.

4, C

(e) Using parts (b), (c), and (d), we have $\dim \ker(D) = 2$, $\dim \text{im}(G) = 3$ and $\ker(D) \cap \text{im}(G) = \{0\}$, hence

unseen ↓

$$\dim(\ker(D) + \text{im}(G)) = 2 + 3 = 5 = \dim(F^5),$$

so $F^5 = \ker(D) + \text{im}(G)$. By rank-nullity, $\text{rk}(D) = 5 - \dim \ker(D) = 3$.

2, B

(f) Assume $2 = 0$ in F . In this case, the vector $(1, 1, 1, 1, 0)$ satisfies the conditions of the hint of (d), and indeed we can set $s = 1, t = 0, u = 1, r = 0$, and we get $G(1, 0, 1, 0)^t = (1, 1, 1, 1, 0)$ (using that $1 = -1$ in F , as a consequence of $2 = 0$).

unseen ↓

2, D

(Total: 20 marks)

2.

seen ↓

(a) Linearity: for $p, q \in V$ and $\lambda \in F$,

$$\begin{aligned} J(p+q)(x) &= (p+q)(-x) = p(-x) + q(-x) = (Jp)(x) + (Jq)(x), \\ J(\lambda p)(x) &= (\lambda p)(-x) = \lambda p(-x) = \lambda(Jp)(x). \end{aligned}$$

So J is linear. On the basis B ,

$$J(1) = 1, \quad J(x) = -x, \quad J(x^2) = x^2, \quad J(x^3) = -x^3,$$

hence $[J]_B = \text{diag}(1, -1, 1, -1)$.

3, A

(b) Since $V^+ = \ker(I - J)$ and $V^- = \ker(I + J)$, both V^+ and V^- are subspaces of V .

seen ↓

Write $p(x) = a + bx + cx^2 + dx^3 \in V$. Then

$$p(-x) = a - bx + cx^2 - dx^3.$$

Case 1: $2 \neq 0$ in F . The condition $p(-x) = p(x)$ gives $2b = 2d = 0$, hence $b = d = 0$, so

$$V^+ = \text{span}\{1, x^2\}.$$

The condition $p(-x) = -p(x)$ gives $2a = 2c = 0$, hence $a = c = 0$, so

$$V^- = \text{span}\{x, x^3\}.$$

In particular $V^+ \cap V^- = \{0\}$, so $\dim(V^+ \cap V^-) = 0$.

Case 2: $2 = 0$ in F . Then $-x = x$, so $p(-x) = p(x)$ for every polynomial p , and also $-p(x) = p(x)$. Hence $V^+ = V^- = V$, so $V^+ \cap V^- = V$ and $\dim(V^+ \cap V^-) = 4$.

7, A

(c) On $B = (1, x, x^2, x^3)$,

$$D(1) = 0, \quad D(x) = 1, \quad D(x^2) = 2x, \quad D(x^3) = 3x^2,$$

so

$$[D]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

seen ↓

2, B

(d) For $p \in V$,

$$(DJp)(x) = \frac{d}{dx}(p(-x)) = -p'(-x) = -(JDp)(x),$$

so $DJ = -JD$. If $p \in V^+$ then $Jp = p$, hence $J(Dp) = -D(Jp) = -Dp$, so $Dp \in V^-$. Similarly, if $p \in V^-$ then $Jp = -p$, so $J(Dp) = -D(Jp) = Dp$, hence $Dp \in V^+$.

unseen ↓

4, C

(e) First suppose $2 \neq 0$ in F . Then $V^+ = \text{span}\{1, x^2\}$ and $V^- = \text{span}\{x, x^3\}$ as in part (b). We have $D(1) = 0$ and $D(x^2) = 2x$, so $\text{rk}(D|_{V^+}) = 1$. Also $D(x) = 1$ and $D(x^3) = 3x^2$, so

unseen ↓

$$\text{rk}(D|_{V^-}) = \begin{cases} 2, & \text{if } 3 \neq 0, \\ 1, & \text{if } 3 = 0. \end{cases}$$

If $2 = 0$ in F : then $V^+ = V^- = V$, so $D|_{V^+} = D|_{V^-} = D : V \rightarrow V$. In this case $D(1) = 0$, $D(x) = 1$, $D(x^2) = 0$ and $D(x^3) = x^2$, so $\text{im}(D) = \langle 1, x^2 \rangle$ and $\text{rk}(D) = 2$. Hence $\text{rk}(D|_{V^+}) = \text{rk}(D|_{V^-}) = 2$.

4, D

(Total: 20 marks)

3. (a) (i) By direct computations, we see that the characteristic polynomial of A is

seen ↓

$$\det(tI_3 - A) = \det \begin{pmatrix} t-3 & 1 & 1 \\ 1 & t-3 & 1 \\ 1 & 1 & t-3 \end{pmatrix} = (t-4)^2(t-1).$$

Hence the eigenvalues of A are 1 and 4.

3, A

(ii) The eigenspace for 1 is

seen ↓

$$E_1 = \ker(I_3 - A) = \ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \text{Span}(v_1)$$

with $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The eigenspace for 4 is

$$E_4 = \ker(4I_3 - A) = \ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \text{Span}(v_2, v_3).$$

with $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. We have learned that v_1 must be orthogonal to both v_2 and v_3 for being in different eigenspaces of the symmetric matrix A . By the Gram-Schmidt process, we find an orthogonal basis of E_4 consists of $w_2 = v_2$ and

$$w_3 = v_3 - \frac{v_3 \cdot w_2}{\|w_2\|^2} w_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}.$$

Thus $u_1 = v_1/\|v_1\|$, $u_2 = w_2/\|w_2\|$ and $u_3 = w_3/\|w_3\|$ form an orthonormal basis. We let

$$Q = (u_1 \mid u_2 \mid u_3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

Then Q is an orthogonal matrix, and we have $A = QDQ^{-1}$ with $D = \text{diag}(1, 4, 4)$.

4, A

(iii) We let

seen ↓

$$B = Q(\text{diag}(1, 2, 2)) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{2} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\sqrt{2} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{4}{\sqrt{6}} \end{pmatrix}.$$

Then $BB^T = Q(\text{diag}(1, 2, 2))^2Q^T = QDQ^{-1} = A$.

3, B

(b) (i) Since $Z = A + iB$ is unitary, we have

unseen ↓

$$I_n = \overline{Z^T}Z = (A^T - iB^T)(A + iB) = (A^T A + B^T B) + i(A^T B - B^T A).$$

Hence $I_n = A^T A + B^T B$ and $0 = A^T B - B^T A$. Thus we have

$$M^T M = \begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} A^T A + B^T B & -A^T B + B^T A \\ -B^T A + A^T B & B^T B + A^T A \end{pmatrix} = I_{2n}.$$

3, A

(ii) We have

unseen ↓

$$JM = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} B & A \\ -A & B \end{pmatrix}$$

and

$$MJ = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} B & A \\ -A & B \end{pmatrix}.$$

This finishes the proof.

3, C

(iii) We write $Q = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$. We have

unseen ↓

$$QJ = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} -C & A \\ -D & B \end{pmatrix}$$

and

$$JQ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} B & D \\ -A & -C \end{pmatrix}.$$

We conclude that $D = A$ and $C = -B$. Thus $Q = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. Since Q is orthogonal, we have $Q^T Q = I_n$, and thus

$$I_{2n} = Q^T Q = \begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} A^T A + B^T B & -A^T B + B^T A \\ -B^T A + A^T B & B^T B + A^T A \end{pmatrix}.$$

Hence $A^T A + B^T B = I_n$ and $A^T B - B^T A = 0$. Let $Z = A + iB$. Then we see that

$$\overline{Z^T} Z = (A^T - iB^T)(A + iB) = (A^T A + B^T B) + i(A^T B - B^T A) = I_n.$$

By our definition, it means that Z is unitary. Thus Q has property (P).

4, D

(Total: 20 marks)

4. (a) (i) The decomposition is found in the standard way by taking elements and following through and is

seen ↓

$$\sigma = (1357)(246)(8\ 10)(9\ 11).$$

4, A

- (ii) By decomposing the cycles we get that

$$\sigma = (13)(35)(57)(24)(46)(8\ 10)(9\ 11)$$

2, A

- (iii) The sign of σ is $(-1)^7 = -1$.

2, A

- (b) (i) We check the three axioms of an equivalence relation. Reflexive $x = gxg^{-1}$ for $g = e_G$, so $x \sim x$. Symmetric, clearly if there is $g \in G$ such that $gxg^{-1} = y$, then $y = g^{-1}xg$ so if $x \sim y$ then $y \sim x$. If $z \in G$ and $y \sim z$, then there is $h \in G$ such that $hyh^{-1} = z$, then $hgx(hg)^{-1} = hgxg^{-1}h^{-1} = hyh^{-1} = z$. Therefore $x \sim z$.

3, B

- (ii) Clearly it suffices to insert $g^{-1}g$ between each repetition of x in the power x^i

$$gx^i g^{-1} = \underbrace{gxg^{-1}gxg^{-1} \cdots gxg^{-1}}_{i \text{ times}} = (gxg^{-1})^i.$$

2, B

- (c) We prove the hint with the following useful formula for conjugation of permutations. If $\sigma, \tau \in S_n$, then:

unseen ↓

$$\sigma(i) = j \Rightarrow (\tau\sigma\tau^{-1})(\tau(i)) = \tau(j).$$

This proves the statement in the hint.

3, C

Now observe that a cycle decomposition

$$\sigma = (a_{1,1} \cdots a_{1,k_1}) \cdots (a_{m,1} \cdots a_{m,k_m}), \quad (1)$$

is characterised by the property that $\sigma(a_{i,j}) = \begin{cases} a_{i,j+1}, & \text{if } 1 \leq j < k_i, \\ a_{i,1}, & \text{otherwise.} \end{cases}$

As a result of the preceding formulas, we conclude, extending the hint to products of cycles, that

$$\tau\sigma\tau^{-1} = (\tau(a_{1,1}) \cdots \tau(a_{1,k_1})) \cdots (\tau(a_{m,1}) \cdots \tau(a_{m,k_m})), \quad (2)$$

which is a decomposition into cycles. Moreover, the original cycle decomposition (1) is disjoint if and only if (2) is (because τ is bijective); also, the cycle lengths k_i are preserved. So we conclude that, if σ has $c(\sigma) = [k_1, \dots, k_m]$, then $\tau\sigma\tau^{-1}$ has $c(\tau\sigma\tau^{-1}) = [k_1, \dots, k_m]$ as well. Conversely, given any σ with $c(\sigma) = [k_1, \dots, k_m]$, consider a disjoint cycle decomposition as in (1). If we let $\tau(a_{i,j}) := k_1 + \cdots + k_{i-1} + j$, then we see that

$$\tau\sigma\tau^{-1} = (1, \dots, k_1)(k_1 + 1, \dots, k_1 + k_2) \cdots (k_1 + \cdots + k_{m-1} + 1, \dots, n).$$

So all such σ are conjugate to a single element, and by transitivity of the conjugation relation, they are all conjugate.

4, D

(Total: 20 marks)

Total A marks: 34 of 32 marks

Total B marks: 18 of 20 marks

Total C marks: 14 of 12 marks

Total D marks: 14 of 16 marks

Solutions

1.

seen ↓

(a) From $D(a, b, c, d, e)^t = (a + e - d, b - a, c - b - e, d - c)^t$, we get

$$[D] = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

3, A

(b) Solve $D(a, b, c, d, e)^t = 0$:

seen ↓

$$\begin{aligned} b - a = 0 &\Rightarrow b = a, & d - c = 0 &\Rightarrow d = c, \\ a + e - d = 0 &\Rightarrow c = d = a + e. \end{aligned}$$

Then the third equation is automatic: $c - b - e = (a + e) - a - e = 0$. Hence

$$\ker(D) = \{(a, a, a + e, a + e, e)^t : a, e \in F\} = \text{span}\{(1, 1, 1, 1, 0)^t, (0, 0, 1, 1, 1)^t\},$$

so $\dim \ker(D) = 2$.

6, B

(c) From

seen ↓

$$G(s, t, u, r)^t = (s - t, t - u, u - r, r - s, s - u)^t$$

we obtain

$$[G] = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Now $G(s, t, u, r)^t = 0$ implies $s = t = u = r$, so $\ker(G) = \text{span}\{(1, 1, 1, 1)^t\}$ and, by the rank-nullity theorem, $\dim(\text{im}(G)) = 4 - 1 = 3$.

3, A

(d) Let $v = (a, b, c, d, e)^t \in \text{im}(G)$. Then $v = G(s, t, u, r)^t$ for some $s, t, u, r \in F$, so

unseen ↓

$$a + b + c + d = (s - t) + (t - u) + (u - r) + (r - s) = 0, \quad e = s - u = (s - t) + (t - u) = a + b.$$

Now suppose $v \in \ker(D)$ as well. Using the description of $\ker(D)$ from part (b), we may write

$$v = (a, a, a + e, a + e, e)^t.$$

The condition $a + b + c + d = 0$ becomes $4a + 2e = 0$, and the condition $e = a + b$ becomes $e = 2a$. Substituting gives $8a = 0$. If 2 is invertible in F (as assumed in the question), then $a = 0$, hence $e = 0$ and therefore $v = 0$. So $\ker(D) \cap \text{im}(G) = \{0\}$.

4, C

(e) Using parts (b), (c), and (d), we have $\dim \ker(D) = 2$, $\dim \text{im}(G) = 3$ and $\ker(D) \cap \text{im}(G) = \{0\}$, hence

unseen ↓

$$\dim(\ker(D) + \text{im}(G)) = 2 + 3 = 5 = \dim(F^5),$$

so $F^5 = \ker(D) + \text{im}(G)$. By rank-nullity, $\text{rk}(D) = 5 - \dim \ker(D) = 3$.

2, B

(f) Assume $2 = 0$ in F . In this case, the vector $(1, 1, 1, 1, 0)$ satisfies the conditions of the hint of (d), and indeed we can set $s = 1, t = 0, u = 1, r = 0$, and we get $G(1, 0, 1, 0)^t = (1, 1, 1, 1, 0)$ (using that $1 = -1$ in F , as a consequence of $2 = 0$).

unseen ↓

2, D

(Total: 20 marks)

2.

seen ↓

(a) Linearity: for $p, q \in V$ and $\lambda \in F$,

$$\begin{aligned} J(p+q)(x) &= (p+q)(-x) = p(-x) + q(-x) = (Jp)(x) + (Jq)(x), \\ J(\lambda p)(x) &= (\lambda p)(-x) = \lambda p(-x) = \lambda(Jp)(x). \end{aligned}$$

So J is linear. On the basis B ,

$$J(1) = 1, \quad J(x) = -x, \quad J(x^2) = x^2, \quad J(x^3) = -x^3,$$

hence $[J]_B = \text{diag}(1, -1, 1, -1)$.

3, A

(b) Since $V^+ = \ker(I - J)$ and $V^- = \ker(I + J)$, both V^+ and V^- are subspaces of V .

seen ↓

Write $p(x) = a + bx + cx^2 + dx^3 \in V$. Then

$$p(-x) = a - bx + cx^2 - dx^3.$$

Case 1: $2 \neq 0$ in F . The condition $p(-x) = p(x)$ gives $2b = 2d = 0$, hence $b = d = 0$, so

$$V^+ = \text{span}\{1, x^2\}.$$

The condition $p(-x) = -p(x)$ gives $2a = 2c = 0$, hence $a = c = 0$, so

$$V^- = \text{span}\{x, x^3\}.$$

In particular $V^+ \cap V^- = \{0\}$, so $\dim(V^+ \cap V^-) = 0$.

Case 2: $2 = 0$ in F . Then $-x = x$, so $p(-x) = p(x)$ for every polynomial p , and also $-p(x) = p(x)$. Hence $V^+ = V^- = V$, so $V^+ \cap V^- = V$ and $\dim(V^+ \cap V^-) = 4$.

7, A

(c) On $B = (1, x, x^2, x^3)$,

$$D(1) = 0, \quad D(x) = 1, \quad D(x^2) = 2x, \quad D(x^3) = 3x^2,$$

so

$$[D]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2, B

(d) For $p \in V$,

$$(DJp)(x) = \frac{d}{dx}(p(-x)) = -p'(-x) = -(JDp)(x),$$

so $DJ = -JD$. If $p \in V^+$ then $Jp = p$, hence $J(Dp) = -D(Jp) = -Dp$, so $Dp \in V^-$. Similarly, if $p \in V^-$ then $Jp = -p$, so $J(Dp) = -D(Jp) = Dp$, hence $Dp \in V^+$.

4, C

(e) First suppose $2 \neq 0$ in F . Then $V^+ = \text{span}\{1, x^2\}$ and $V^- = \text{span}\{x, x^3\}$ as in part (b). We have $D(1) = 0$ and $D(x^2) = 2x$, so $\text{rk}(D|_{V^+}) = 1$. Also $D(x) = 1$ and $D(x^3) = 3x^2$, so

unseen ↓

$$\text{rk}(D|_{V^-}) = \begin{cases} 2, & \text{if } 3 \neq 0, \\ 1, & \text{if } 3 = 0. \end{cases}$$

If $2 = 0$ in F : then $V^+ = V^- = V$, so $D|_{V^+} = D|_{V^-} = D : V \rightarrow V$. In this case $D(1) = 0$, $D(x) = 1$, $D(x^2) = 0$ and $D(x^3) = x^2$, so $\text{im}(D) = \langle 1, x^2 \rangle$ and $\text{rk}(D) = 2$. Hence $\text{rk}(D|_{V^+}) = \text{rk}(D|_{V^-}) = 2$.

4, D

(Total: 20 marks)

3. (a) (i) By direct computations, we see that the characteristic polynomial of A is

seen ↓

$$\det(tI_3 - A) = \det \begin{pmatrix} t-3 & 1 & 1 \\ 1 & t-3 & 1 \\ 1 & 1 & t-3 \end{pmatrix} = (t-4)^2(t-1).$$

Hence the eigenvalues of A are 1 and 4.

3, A

(ii) The eigenspace for 1 is

seen ↓

$$E_1 = \ker(I_3 - A) = \ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \text{Span}(v_1)$$

with $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The eigenspace for 4 is

$$E_4 = \ker(4I_3 - A) = \ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \text{Span}(v_2, v_3).$$

with $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. We have learned that v_1 must be orthogonal to both v_2 and v_3 for being in different eigenspaces of the symmetric matrix A . By the Gram-Schmidt process, we find an orthogonal basis of E_4 consists of $w_2 = v_2$ and

$$w_3 = v_3 - \frac{v_3 \cdot w_2}{\|w_2\|^2} w_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}.$$

Thus $u_1 = v_1/\|v_1\|$, $u_2 = w_2/\|w_2\|$ and $u_3 = w_3/\|w_3\|$ form an orthonormal basis. We let

$$Q = (u_1 \mid u_2 \mid u_3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

Then Q is an orthogonal matrix, and we have $A = QDQ^{-1}$ with $D = \text{diag}(1, 4, 4)$.

4, A

(iii) We let

seen ↓

$$B = Q(\text{diag}(1, 2, 2)) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{2} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\sqrt{2} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{4}{\sqrt{6}} \end{pmatrix}.$$

Then $BB^T = Q(\text{diag}(1, 2, 2))^2Q^T = QDQ^{-1} = A$.

3, B

(b) (i) Since $Z = A + iB$ is unitary, we have

unseen ↓

$$I_n = \overline{Z^T}Z = (A^T - iB^T)(A + iB) = (A^T A + B^T B) + i(A^T B - B^T A).$$

Hence $I_n = A^T A + B^T B$ and $0 = A^T B - B^T A$. Thus we have

$$M^T M = \begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} A^T A + B^T B & -A^T B + B^T A \\ -B^T A + A^T B & B^T B + A^T A \end{pmatrix} = I_{2n}.$$

3, A

(ii) We have

unseen ↓

$$JM = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} B & A \\ -A & B \end{pmatrix}$$

and

$$MJ = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} B & A \\ -A & B \end{pmatrix}.$$

This finishes the proof.

3, C

(iii) We write $Q = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$. We have

unseen ↓

$$QJ = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = \begin{pmatrix} -C & A \\ -D & B \end{pmatrix}$$

and

$$JQ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} B & D \\ -A & -C \end{pmatrix}.$$

We conclude that $D = A$ and $C = -B$. Thus $Q = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. Since Q is orthogonal, we have $Q^T Q = I_{2n}$, and thus

$$I_{2n} = Q^T Q = \begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} A^T A + B^T B & -A^T B + B^T A \\ -B^T A + A^T B & B^T B + A^T A \end{pmatrix}.$$

Hence $A^T A + B^T B = I_n$ and $A^T B - B^T A = 0$. Let $Z = A + iB$. Then we see that

$$\overline{Z^T} Z = (A^T - iB^T)(A + iB) = (A^T A + B^T B) + i(A^T B - B^T A) = I_n.$$

By our definition, it means that Z is unitary. Thus Q has property (P).

4, D

(Total: 20 marks)

4. (a) (i) The decomposition is found in the standard way by taking elements and following through and is

seen ↓

$$\sigma = (1357)(246)(8\ 10)(9\ 11).$$

4, A

- (ii) By decomposing the cycles we get that

$$\sigma = (13)(35)(57)(24)(46)(8\ 10)(9\ 11)$$

2, A

- (iii) The sign of σ is $(-1)^7 = -1$.

2, A

- (b) (i) We check the three axioms of an equivalence relation. Reflexivity follows because $gxg^{-1} = x$ for $g = e_G$, so $x \sim x$. Symmetry follows because, if there is $g \in G$ such that $gxg^{-1} = y$, then $g^{-1}yg = x$ so if $x \sim y$ then $y \sim x$. Finally, for transitivity, suppose that $z \in G$ and $y \sim z$. Then there are $g, h \in G$ such that $gxg^{-1} = y$ and $hyh^{-1} = z$, so $hgx(hg)^{-1} = hgxg^{-1}h^{-1} = hyh^{-1} = z$. Therefore $x \sim z$.

3, B

- (ii) Clearly it suffices to insert $g^{-1}g$ between each repetition of x in the power x^i

$$gx^i g^{-1} = \underbrace{gxg^{-1}gxg^{-1} \cdots gxg^{-1}}_{i \text{ times}} = (gxg^{-1})^i.$$

2, B

- (c) We prove the hint with the following useful formula for conjugation of permutations. If $\sigma, \tau \in S_n$, then:

unseen ↓

$$\sigma(i) = j \Rightarrow (\tau\sigma\tau^{-1})(\tau(i)) = \tau(j).$$

This proves the statement in the hint.

3, C

Now observe that a cycle decomposition

$$\sigma = (a_{1,1} \cdots a_{1,k_1}) \cdots (a_{m,1} \cdots a_{m,k_m}), \quad (1)$$

is characterised by the property that $\sigma(a_{i,j}) = \begin{cases} a_{i,j+1}, & \text{if } 1 \leq j < k_i, \\ a_{i,1}, & \text{otherwise.} \end{cases}$

As a result of the preceding formulas, we conclude, extending the hint to products of cycles, that

$$\tau\sigma\tau^{-1} = (\tau(a_{1,1}) \cdots \tau(a_{1,k_1})) \cdots (\tau(a_{m,1}) \cdots \tau(a_{m,k_m})), \quad (2)$$

which is a decomposition into cycles. Moreover, the original cycle decomposition (1) is disjoint if and only if (2) is (because τ is bijective); also, the cycle lengths k_i are preserved. So we conclude that, if σ has $c(\sigma) = [k_1, \dots, k_m]$, then $\tau\sigma\tau^{-1}$ has $c(\tau\sigma\tau^{-1}) = [k_1, \dots, k_m]$ as well. Conversely, given any σ with $c(\sigma) = [k_1, \dots, k_m]$, consider a disjoint cycle decomposition as in (1). If we let $\tau(a_{i,j}) := k_1 + \cdots + k_{i-1} + j$, then we see that

$$\tau\sigma\tau^{-1} = (1, \dots, k_1)(k_1 + 1, \dots, k_1 + k_2) \cdots (k_1 + \dots + k_{m-1} + 1, \dots, n).$$

So all such σ are conjugate to a single element, and by transitivity of the conjugation relation, they are all conjugate.

4, D

(Total: 20 marks)

Total A marks: 34 of 32 marks

Total B marks: 18 of 20 marks

Total C marks: 14 of 12 marks

Total D marks: 14 of 16 marks