

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Linear Algebra and Groups

Date: Tuesday, 4 May 2021

Time: 09:00 to 12:00

Time Allowed: 3 hours

Upload Time Allowed: 45 minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) For which values of $\alpha \in \mathbb{R}$ does the system of equations

$$\left. \begin{aligned} 2x - y - 3z &= 3 \\ \alpha x - 4y + 2z &= 5\alpha \\ -3x + 6y + 2z &= -5 \end{aligned} \right\} \text{ for } x, y, z \in \mathbb{R}$$

have:

- (i) no solutions?
- (ii) exactly one solution?
- (iii) infinitely many solutions?

In parts (ii) and (iii) you should give the solutions. (10 marks)

- (b) Suppose V is a vector space over a field F . Suppose $0_V \notin W$ and $W \subseteq V$. Is W a subspace of V ? Give full details of the argument, assuming only axioms of a vector space/field.

(2 marks)

- (c) Let S be the set of solutions to the system in (a) (where $\alpha \in \mathbb{R}$ is a constant). Which of the following sets are subspaces of $\mathbb{R}^3$? Does the answer depend on the value of α ? Justify your answer fully.

- (i) S .

- (ii)

$$\begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix} + S = \{v \in \mathbb{R}^3 : v = \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix} + s \text{ for some } s \in S\}.$$

- (iii)

$$\begin{pmatrix} -\frac{13}{9} \\ -\frac{1}{9} \\ 0 \end{pmatrix} + S = \{v \in \mathbb{R}^3 : v = \begin{pmatrix} -\frac{13}{9} \\ -\frac{1}{9} \\ 0 \end{pmatrix} + s \text{ for some } s \in S\}.$$

(8 marks)

(Total: 20 marks)

2. Let V be the $\mathbb{R}$ -vector space $\mathbb{R}^4$. Let

$$W = \left\{ \begin{pmatrix} a \\ a+b \\ -3b \\ 2b+a \end{pmatrix} \in \mathbb{R}^4 : a, b \in \mathbb{R} \right\},$$

$$B = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} \right), \quad \text{and } C = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} \right).$$

(a) Show that W is a subspace of V . (3 marks)

(b) Show that B is a basis for V and find $\left[\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \right]_B$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$. (5 marks)

(c) Let $T : V \rightarrow W$ be the linear map such that:

$$T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2z \\ 3y + z - 3w \\ -9y + 3z + 9w \\ 6y - 6w \end{pmatrix}.$$

(i) Find $\text{Im}(T)$ and $\dim(\text{Im}(T))$.

(ii) Find ${}_C[T]_B$. (5 marks)

(d) Let T, V, W be as above. Let V^* be the $\mathbb{R}$ -vector space of linear transformations from V to V . And let U^* be the $\mathbb{R}$ -vector space of linear transformations from V to W . Let

$$T^* : V^* \rightarrow U^* \text{ be such that } T^*(f) = T \circ f.$$

Show that

(i) T^* is a linear transformation.

(ii) If B^* is a basis for V^* and C^* is a basis for U^* determine the rank of ${}_{C^*}[T^*]_{B^*}$.

Any results from lectures used must be clearly stated. (7 marks)

(Total: 20 marks)

3. Let V be an $\mathbb{R}$ -vector space and let $b_1, \dots, b_n$ be a basis for V with $n \geq 2$. We say an *idempotent* is a linear transformation $T : V \rightarrow V$ such that $T^2 = T \circ T = T$.

(a) Denoting a general element of V by $v = \alpha_1 b_1 + \dots + \alpha_n b_n$ (for $\alpha_1, \dots, \alpha_n \in \mathbb{R}$), which of the following maps are:

- * Linear transformations?
- * Idempotents?

Give clear justification in all cases.

- i. $S_0 : V \rightarrow V$, where $S_0(v) = 0_V$.
- ii. $S_1 : V \rightarrow V$, where $S_1(v) = b_1$.
- iii. $S_2 : V \rightarrow V$, where $S_2(v) = \alpha_1 b_1 + \alpha_2 b_2$.
- iv. $S_3 = (S_2)^2 = S_2 \circ S_2$.
- v. $S_4 : V \rightarrow V$, where $S_4(v) = \alpha_2 b_1 + \alpha_1 b_2$.

(11 marks)

(b) Show that if $T : V \rightarrow V$ and $S : V \rightarrow V$ are linear transformations, then $S \circ T$ is a linear transformation. (2 marks)

(c) Suppose $T : V \rightarrow V$ and $S : V \rightarrow V$ are idempotent. Is $S \circ T$ idempotent? *Justify your answer fully giving a proof or counterexample.* (4 marks)

(d) When can you construct an idempotent linear transformation $T : V \rightarrow V$ such that $\text{Im}(T) = \ker(T)$. *Justify your answer fully, any results used from lectures must be clearly stated.* (3 marks)

(Total: 20 marks)

4. (a) The sequences of real numbers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ satisfy the relations:

$$a_{n+1} = 3a_n + 2b_n \text{ and } b_{n+1} = -4a_n - 3b_n \text{ for all } n \geq 1.$$

- (i) Write down a matrix A with the property that

$$A \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}$$

and compute its eigenvalues and eigenvectors (over $\mathbb{R}$), showing the details of your computations.

(6 marks)

- (ii) Hence, or otherwise, give general expressions for a_n and b_n in terms of n, a_1 and b_1 .

(3 marks)

- (iii) Which values for a_1 and b_1 imply that both $a_{17} = 3$ and $b_{18} = 12$?

(1 mark)

- (b) (i) Suppose $v_1, \dots, v_r \in \mathbb{R}^n$ is an orthogonal set of non-zero vectors. Prove that for all $\alpha_1, \dots, \alpha_r \in \mathbb{R}$ we have

$$\left\| \sum_{i=1}^r \alpha_i v_i \right\| = \left(\sum_{i=1}^r \alpha_i^2 \|v_i\|^2 \right)^{\frac{1}{2}}.$$

(2 marks)

- (ii) Suppose $A \in M_n(\mathbb{R})$ is symmetric. Let α be the maximum value of $|\lambda|$ for all eigenvalues λ of A .

- Using (b)(i), show that $\|Av\| \leq \alpha\|v\|$ for all $v \in \mathbb{R}^n$ and there is some non-zero v for which we have equality. Clearly state any results from the notes which you require.
- Deduce that $\sup\{\|Av\| : \|v\| = 1\} = \alpha$.
- State and prove a similar result for $\inf\{\|Av\| : \|v\| = 1\}$.

(8 marks)

(Total: 20 marks)

5. (a) Let $n \geq 2$ and let F be a field. Suppose $A \in M_n(F)$ has exactly n non-zero entries $x_1, \dots, x_n$, one in each row and one in each column of A .
- (i) Prove that $\det(A)$ is equal to $x_1 x_2 \dots x_n$ or $-x_1 x_2 \dots x_n$.
- (ii) One of the zero entries in A is changed to 1, resulting in the matrix B . Express $\det(B)$ in terms of $\det(A)$, explaining your answer. (6 marks)
- (b) Suppose $C \in M_4(F)$ and $\lambda \in F$ is an eigenvalue of C such that the eigenspace E_λ has dimension 2. Prove that $\chi_C(x) = (x - \lambda)^2 g(x)$ for some polynomial $g(x)$ with coefficients in F . (You may quote results from the notes, but you should not quote general results which immediately imply what is to be proved.) (6 marks)
- (c) Let G be the symmetric group S_6 and $H = \{h \in G : \{h(1), h(2)\} = \{1, 2\}\}$.
- (i) Show that H is a subgroup of G and determine $|H|$, explaining your answer. (3 marks)
- (ii) Prove that for $g, k \in G$ we have
- $$gH = kH \Leftrightarrow \{g(1), g(2)\} = \{k(1), k(2)\}.$$
- (3 marks)
- (iii) Using (c)(i) and (ii), compute the index of H in G in two different ways. (2 marks)
- (Throughout the question, results from the notes may be used if clearly stated.)

(Total: 20 marks)

6. (a) Suppose $(G, \cdot)$ is a group and let $H = \{g^2 : g \in G\}$. Three of the following four statements are true and one is not true in general. Prove the ones which are true and give, with explanation, a counterexample for the other one.

- (i) If G is abelian, then H is a subgroup of G .
- (ii) If G is finite and $|G|$ is odd, then $H = G$.
- (iii) If $n \geq 4$ and G is the symmetric group S_n , then $H = G$.
- (iv) If G is the dihedral group D_{2n} , then H is a subgroup of G .

(8 marks)

(b) Consider the following elements of S_9 :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 & 1 \end{pmatrix}; g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

- (i) Write $f, g, f^{-1}, g^3 f g f^2$ in disjoint cycle form and state the orders of these. (4 marks)
- (ii) What are the cycle shapes of elements of order 3 in S_9 ? How many elements of order 3 are there in S_9 ? (You may leave your answers expressed in terms of binomial coefficients.) (4 marks)

(c) Suppose G is a finite group. Prove that the number of elements of order 5 in G is divisible by 4. State and prove a more general result about the number of elements of order p in G , where p is a prime number. (4 marks)

(Throughout the question, results from the notes may be used if clearly stated.)

(Total: 20 marks)

1. (a) For which values of $\alpha \in \mathbb{R}$ does the system of equations

$$\left. \begin{aligned} 2x - y - 3z &= 3 \\ \alpha x - 4y + 2z &= 5\alpha \\ -3x + 6y + 2z &= -5 \end{aligned} \right\} \text{ for } x, y, z \in \mathbb{R}$$

have:

- (i) no solutions?
- (ii) exactly one solution?
- (iii) infinitely many solutions?

In parts (ii) and (iii) you should give the solutions.

(10 marks)

$$\left(\begin{array}{ccc|c} 2 & -1 & -3 & 3 \\ \alpha & -4 & 2 & 5\alpha \\ -3 & 6 & 2 & -5 \end{array} \right) \xrightarrow{r_1 \rightarrow -r_1, r_2} \left(\begin{array}{ccc|c} 1 & -5 & 1 & 2 \\ \alpha & -4 & 2 & 5\alpha \\ -3 & 6 & 2 & -5 \end{array} \right)$$

(1 mark for correct augmented matrix) (1A).

$$\xrightarrow{r_3 \rightarrow r_3 + 3r_1} \left(\begin{array}{ccc|c} 1 & -5 & 1 & 2 \\ \alpha & -4 & 2 & 5\alpha \\ 0 & -9 & 5 & 1 \end{array} \right)$$

$$\xrightarrow{r_3 \rightarrow \frac{r_3}{-9}} \left(\begin{array}{ccc|c} 1 & -5 & 1 & 2 \\ \alpha & -4 & 2 & 5\alpha \\ 0 & 1 & -5/9 & -1/9 \end{array} \right)$$

$$\xrightarrow{r_1 \rightarrow r_1 + 5r_3} \left(\begin{array}{ccc|c} 1 & 0 & -16/9 & 13/9 \\ \alpha & -4 & 2 & 5\alpha \\ 0 & 1 & -5/9 & -1/9 \end{array} \right)$$

$$\xrightarrow{r_2 \rightarrow r_2 + 4r_3} \left(\begin{array}{ccc|c} 1 & 0 & -16/9 & 13/9 \\ \alpha & 0 & -2/9 & \frac{5\alpha - 4}{9} \\ 0 & 1 & -5/9 & -1/9 \end{array} \right)$$

$$\xrightarrow{r_2 - \alpha r_1} \begin{pmatrix} 1 & 0 & -16/9 & 13/9 \\ 0 & 0 & \frac{16\alpha-2}{9} & \frac{32\alpha-4}{9} \\ 0 & 1 & -5/9 & -1/9 \end{pmatrix} \quad (3 \text{ marks A})$$

If $16\alpha - 2 \neq 0$ then

$$\xrightarrow{r_2 \rightarrow r_2 / (16\alpha-2)} \begin{pmatrix} 1 & 0 & -16/9 & 13/9 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & -5/9 & -1/9 \end{pmatrix}$$

$$\xrightarrow{r_1 \rightarrow r_1 + 16/9 r_2} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & -5/9 & -1/9 \end{pmatrix}$$

$$\xrightarrow{r_3 \rightarrow r_3 + 5/9 r_2} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad (2 \text{ marks A})$$

If $16\alpha - 2 = 0$ then.

$$\longrightarrow \begin{pmatrix} 1 & 0 & -16/9 & 13/9 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -5/9 & -1/9 \end{pmatrix} \quad \begin{aligned} x - \frac{16}{9}z &= \frac{13}{9} \\ y - \frac{5}{9}z &= -\frac{1}{9} \end{aligned}$$

The solution set is the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16/9 \\ 5/9 \\ 1 \end{pmatrix} t + \begin{pmatrix} 13/9 \\ -1/9 \\ 0 \end{pmatrix} \quad t \in \mathbb{R}.$

(1 mark A)

i) This system always has at least one solution. (1 mark A)

ii) This system has one solution when $16\alpha - 2 \neq 0$ in this case the solution

ii) This system has one solution when $16\alpha - 2 \neq 0$ in this case the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} \quad (1 \text{ mark A})$$

iii) This system has infinitely many solutions when $16\alpha - 2 = 0$ in this case the solutions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16/9 \\ 5/9 \\ 1 \end{pmatrix} t + \begin{pmatrix} 13/9 \\ -1/9 \\ 0 \end{pmatrix} \quad t \in \mathbb{R}. \quad (1 \text{ mark A})$$

b) We will prove the contrapositive (i.e. If W is a subspace then $0_V \in W$).

Let W be a subspace.

By S_1 , $\exists w \in W$. by S_3 $0 \cdot w \in W$ (1 mark B)

Now show $0_W = 0_V$

$$\begin{aligned} 0_V &= 0 \cdot w - 0 \cdot w \\ &= 0 \cdot w - (0 - 0) \cdot w \\ &= (0 \cdot w - 0 \cdot w) + 0 \cdot w \\ &= 0_V + 0 \cdot w \\ &= 0 \cdot w. \end{aligned}$$

(1 mark B)

not required.
(add inverse)
($0 - 0 = 0$ in F)
(distributive + asso)
(add inverse)
(add identity).

(c) Let S be the set of solutions to the system in (a). Which of the following sets are subspaces of $\mathbb{R}^3$? Does the answer depend on the value of α ? Justify your answer fully.

(i) S .

c) i) By above $S = \left\{ v \in \mathbb{R}^3 : v = \begin{pmatrix} 16/9 \\ 5/9 \\ 1 \end{pmatrix} t + \begin{pmatrix} 13/9 \\ -1/9 \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}$

Now $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin S$ because if $\begin{pmatrix} 16/9 \\ 5/9 \\ 1 \end{pmatrix} t + \begin{pmatrix} 13/9 \\ -1/9 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

then $t + 0 = 0$ so $t = 0$ but then

$$\begin{pmatrix} 16/9 \\ 5/9 \\ 1 \end{pmatrix} t + \begin{pmatrix} 13/9 \\ -1/9 \\ 0 \end{pmatrix} = \begin{pmatrix} 13/9 \\ -1/9 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (1 \text{ mark B})$$

Now, by part (b) S cannot be a vector space. (1 mark B)

(ii)

$$\begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix} + S = \left\{ v \in \mathbb{R}^3 : v = \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix} + s \text{ for some } s \in S \right\}.$$

ii) Let $S + \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix} = T$

If $16\alpha - 2 \neq 0$ then is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ which is the trivial subspace

If $16\alpha - 2 = 0$ then this is

$$T = \left\{ v \in \mathbb{R}^3 : v = \begin{pmatrix} 16/9 \\ 5/9 \\ 1 \end{pmatrix} t + \begin{pmatrix} 13/9 \\ -1/9 \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}, t \in \mathbb{R} \right\}.$$

Let $t = r + 2$ then $= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$T = \left\{ v \in \mathbb{R}^3 : v = \begin{pmatrix} 16/9 \\ 5/9 \\ 1 \end{pmatrix} r + \begin{pmatrix} 32/9 \\ 10/9 \\ 2 \end{pmatrix} + \begin{pmatrix} 13/9 \\ -1/9 \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}, r \in \mathbb{R} \right\}$$

$$= \left\{ v \in \mathbb{R}^3 : v = \begin{pmatrix} 16/a \\ 5/a \\ 1 \end{pmatrix} r, r \in \mathbb{R} \right\}. \quad (2 \text{ marks } C)$$

$$= \text{span} \left(\begin{pmatrix} 16/a \\ 5/a \\ 1 \end{pmatrix} \right)$$

which is a subspace.

(1 mark B)

(iii)

$$\begin{pmatrix} -\frac{13}{9} \\ -\frac{1}{9} \\ 0 \end{pmatrix} + S = \{v \in \mathbb{R}^3 : v = \begin{pmatrix} -\frac{13}{9} \\ -\frac{1}{9} \\ 0 \end{pmatrix} + s \text{ for some } s \in S\}.$$

$$(iii) \text{ Let } U = S + \begin{pmatrix} -13/a \\ 1/a \\ 0 \end{pmatrix}$$

$$\text{When } 16a - 2 \neq 0 \text{ then } U = \left\{ \begin{pmatrix} 32/a \\ 10/a \\ 2 \end{pmatrix} \right\} \text{ so } \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin U \text{ so by part (b)}$$

U is not a subspace

(1 mark B)

If $16a - 2 = 0$ then

$$U = \left\{ v \in \mathbb{R}^3 : v = \begin{pmatrix} 16/a \\ 5/a \\ 1 \end{pmatrix} t + \begin{pmatrix} 13/a \\ -1/a \\ 0 \end{pmatrix} + \begin{pmatrix} -13/a \\ 1/a \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}.$$

$$= \left\{ v \in \mathbb{R}^3 : v = \begin{pmatrix} 16/a \\ 5/a \\ 1 \end{pmatrix} t, t \in \mathbb{R} \right\}.$$

= T so by part (i) this is a subspace.

(2 marks C)

Part (a) they have seen similar (b) is on a problem sheet
(c) is unseen.

2. Let V be the $\mathbb{R}$ -vector space $\mathbb{R}^4$. Let

$$W = \left\{ \begin{pmatrix} a \\ a+b \\ -3b \\ 2b+a \end{pmatrix} \in \mathbb{R}^4 : a, b \in \mathbb{R} \right\},$$

$$B = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} \right), \quad \text{and } C = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} \right).$$

(a) Show that W is a subspace of V .

(3 marks)

S1: Let $a=b=0$ then $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in W$

S2: Let $v_1, v_2 \in W$ then $v_1 = \begin{pmatrix} a_1 \\ -a_1 + b_1 \\ -3b_1 \\ 2b_1 + a_1 \end{pmatrix}$ $v_2 = \begin{pmatrix} a_2 \\ -a_2 + b_2 \\ -3b_2 \\ 2b_2 + a_2 \end{pmatrix}$ $a_1, a_2, b_1, b_2 \in \mathbb{R}$

$$v_1 + v_2 = \begin{pmatrix} a_1 + a_2 \\ -a_1 + b_1 - a_2 + b_2 \\ -3b_1 - 3b_2 \\ 2b_1 + a_1 + 2b_2 + a_2 \end{pmatrix} = \begin{pmatrix} (a_1 + a_2) \\ -(a_1 + a_2) + (b_1 + b_2) \\ -3(b_1 + b_2) \\ 2(b_1 + b_2) + (a_1 + a_2) \end{pmatrix} \in W \text{ as } a_1 + a_2, b_1 + b_2 \in \mathbb{R}.$$

S3: Let $v \in W$ then $v = \begin{pmatrix} a \\ -a + b \\ -3b \\ 2b + a \end{pmatrix}$, and let $\lambda \in \mathbb{R}$.

$$\lambda v = \begin{pmatrix} \lambda a \\ \lambda(-a + b) \\ \lambda(-3b) \\ \lambda(2b + a) \end{pmatrix} = \begin{pmatrix} \lambda a \\ -\lambda a + \lambda b \\ -3\lambda b \\ 2\lambda b + \lambda a \end{pmatrix} \in W \text{ as } \lambda a, \lambda b \in \mathbb{R}.$$

(3 marks A)

(b) Show that B is a basis for V and find $\begin{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \end{bmatrix}_B$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$. (5 marks)

First note that

$$\bullet \begin{pmatrix} 0 \\ 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\bullet \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Thus letting $v \in V$

$$v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\alpha_3}{3} \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ + \alpha_4 \left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$= \left(\alpha_1 - \frac{2\alpha_3}{3} - \alpha_4 \right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \left(\alpha_2 - \frac{\alpha_3}{3} - \alpha_4 \right) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \left(\frac{2\alpha_3}{3} + \alpha_4 \right) \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} + \left(-\frac{\alpha_3}{3} \right) \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} \in \text{Span } B.$$

So B spans V

(2 marks B)

$$\text{If } \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{then } \lambda_1 + \lambda_3 &= 0 & (1) \\ \lambda_2 + \lambda_3 &= 0 & (2) \\ -3\lambda_4 &= 0 & (3) \\ \lambda_3 + 2\lambda_4 &= 0 & (4) \end{aligned}$$

By (3) $\lambda_4 = 0$, sub into (4) gives $\lambda_3 = 0$, subbing into (1), (2) gives you $\lambda_1 = 0, \lambda_2 = 0$ resp.

Thus $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ so $\mathcal{B}$ is linearly independent.

Also have to use result that 4 spanning vectors in 4-dim space form a basis if result clearly stated. (2 marks A)

From

$$\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} a_1 - 2a_3/3 - a_4 \\ a_2 - a_3/3 - a_4 \\ 2a_3/3 + a_4 \\ -a_3/3 \end{pmatrix}$$

(1 mark A).

(c) Let $T : V \rightarrow W$ such that:

$$T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2z \\ 3y + z - 3w \\ -9y + 3z + 9w \\ 6y - 6w \end{pmatrix}$$

(i) Find $\text{Im}(T)$ and $\dim(\text{Im}(T))$.

(ii) Find ${}_B[T]_C$.

$$\begin{aligned} \text{i) } \text{Im}(T) &= \{w \in W : \exists v \in V \text{ } T(v) = w\} \\ &= \left\{ w \in W : w = \begin{pmatrix} 2z \\ 3y + z - 3w \\ -9y + 3z + 9w \\ 6y - 6z \end{pmatrix} \text{ for } y, z, w \in \mathbb{R} \right\} \\ &= \left\{ w \in W : w = y \begin{pmatrix} 0 \\ 3 \\ -9 \\ 6 \end{pmatrix} + z \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix} + w \begin{pmatrix} 0 \\ -3 \\ 9 \\ -6 \end{pmatrix} \text{ for } y, z, w \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{pmatrix} 0 \\ 3 \\ -9 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 9 \\ -6 \end{pmatrix} \right\} \\ &= \text{span} \left\{ \begin{pmatrix} 0 \\ 3 \\ -9 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right\} \\ &= \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

(1 mark A)

As $\begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix}$ are linearly independent $\dim(\text{Im}(T)) = 2$. (1 mark A)
(So $\text{Im}(T) = W$)

(ii) Find ${}_B[T]_C$

$$\text{Let } b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b_3 = c_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad b_4 = c_2 = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix}$$

Then

$$T(b_1) = 0_V = 0c_1 + 0c_2$$

$$T(b_2) = \begin{pmatrix} 0 \\ 3 \\ -9 \\ 6 \end{pmatrix} = 0c_1 + 3c_2$$

$$T(b_3) = \begin{pmatrix} 0 \\ 3 & -3 \\ -9 & 9 \\ 6 & -6 \end{pmatrix} = 0_V = 0c_1 + 0c_2$$

$$T(b_4) = \begin{pmatrix} -6 \\ 3 + (-3) & -6 \\ -9 & -9 & +18 \\ 6 & -12 \end{pmatrix} = \begin{pmatrix} -6 \\ -6 \\ 0 \\ -6 \end{pmatrix} = -6 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{So } {}_B T_C = \begin{pmatrix} 0 & 0 & 0 & -6 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

(3 marks B).

- (d) Let V^* be the $\mathbb{R}$ -vector space of linear transformations from V to $\mathbb{R}$. And let U^* be the $\mathbb{R}$ -vector space of linear transformations from V to W .

Let

$$T^* : V^* \rightarrow U^* \text{ be such that } T^*(f) = T \circ f$$

Show that

1. T^* is a linear transformation.
2. If B^* is a basis for V^* and C^* is a basis for U^* determine the rank of $B^*[T^*]_{C^*}$.

(i) T^* preserves addition:

$$\text{Let } f_1, f_2 \in V^*, v \in V$$

$$\begin{aligned} (T^*(f_1 + f_2))(v) &= (T \circ (f_1 + f_2))(v) \\ &= T((f_1 + f_2)(v)) \\ &= T(f_1(v) + f_2(v)) \\ &= T(f_1(v)) + T(f_2(v)) \quad \text{as } T \text{ linear} \\ &= T \circ f_1(v) + T \circ f_2(v) \\ &= (T^*(f_1) + T^*(f_2))(v), \end{aligned}$$

As this is true for all $v \in V$ we get.

$$T^*(f_1 + f_2) = T^*(f_1) + T^*(f_2) \quad (2 \text{ marks } \textcircled{D}).$$

T^* preserves scalar mult:

Let $f \in V^*$, $\lambda \in \mathbb{R}$, and let $v \in V$

$$\begin{aligned} T^*(\lambda f)(v) &= (T \circ (\lambda f))(v) \\ &= T(\lambda f(v)) \end{aligned}$$

$$\begin{aligned}
 T^*(\lambda f)(v) &= (I \circ (\lambda f))(v) \\
 &= T(\lambda f(v)) \\
 &= \lambda T(f(v)) & T \text{ linear.} \\
 &= \lambda (T \circ f)(v) \\
 &= \lambda T^*(f)(v)
 \end{aligned}$$

As this is $\forall v \in V$ $T^*(\lambda f) = \lambda T^*(f)$. (2 marks D)

Thus T^* is a linear transformation.

(ii) V^* is isomorphic to the set of 4×4 matrices using $\Phi: V^* \rightarrow M_{4 \times 4}(\mathbb{R})$ $\Phi(f) = [f]_B$
 U^* is isomorphic to the set of 2×4 matrices using $\Phi': U^* \rightarrow M_{2 \times 4}(\mathbb{R})$ $\Phi'(h) = [h]_C$

$$\begin{array}{ccc}
 T^*: & M_{4 \times 4}(\mathbb{R}) & \rightarrow M_{4 \times 2} \\
 & \begin{smallmatrix} \text{S.I.} \\ \mathbb{R}^{16} \end{smallmatrix} & \begin{smallmatrix} \text{S.I.} \\ \mathbb{R}^8 \end{smallmatrix}
 \end{array}$$

Now rank ${}_B[T]_C^*$ is equal to the dimension of $\text{Im } T^*$

$$\begin{array}{ccc}
 \text{Now} & \begin{pmatrix} 0 & 0 & 0 & -6 \\ 0 & 3 & 0 & 0 \end{pmatrix} & \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} -6a_{41} - 6a_{42} & -6a_{43} & -6a_{44} \\ 3a_{21} & 3a_{22} & 3a_{23} & 3a_{24} \end{pmatrix} \\
 2 \times 4 & & 4 \times 4 \qquad \qquad \qquad 2 \times 4
 \end{array}$$

So as $\text{Im}(T^*) = U^*$ we get the rank of ${}_B[T]_C^*$ is $\dim U^* = 8$.

(3 marks D)

(a), (b), (c) seen similar on problem sheets.

(a), (b), (c) seen similar on problem sheets.

(d) unseen - and probably quite challenging.

3. Let V be an $\mathbb{R}$ -vector space and let $\{b_1, \dots, b_n\}$ be a basis for V with $n \geq 2$. We say a linear transformation $T : V \rightarrow V$ is *idempotent* if $T^2 = T \circ T = T$

(a) Let $v \in V$ be of the form $v = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$ which of the following maps are linear transformations? Which of those are idempotent? Give clear justification in all cases.

i. $S_0 : V \rightarrow V, S(v) = 0_V$

LINEAR:

Let $v_1, v_2 \in V, \lambda \in \mathbb{R}$ then

$$S_0(v_1 + \lambda v_2) = 0_V = 0_V + \lambda 0_V = S_0(v_1) + \lambda S_0(v_2)$$

So S_0 preserves addition + scalar mult so is linear.

(2 marks A)

IDEMPOTENT

Let $v \in V$ $S_0(S_0(v)) = S_0(0_V) = 0_V = S_0(v)$ thus $S_0^2 = S_0$ so S_0 is idempotent.

(1 mark C)

ii. $S_1 : V \rightarrow V, S(v) = b_1$

NOT LINEAR: $S_1(b_1) + S_1(b_2) = b_1 + b_1 \neq b_1 = S_1(b_1 + b_2)$

as $b_1 \neq 0_V$ ✓

S_1 doesn't preserve addition.

(1 mark B)

iii. $S_2 : V \rightarrow V, S(v) = \alpha_2 b_1 + \alpha_1 b_2$

LINEAR:

Let $v_1, v_2 \in V, \lambda \in \mathbb{R}$, with $v_1 = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$
 $v_2 = \beta_1 b_1 + \dots + \beta_n b_n$

So $v_1 + \lambda v_2 = (\alpha_1 + \lambda \beta_1) b_1 + \dots + (\alpha_n + \lambda \beta_n) b_n$.

$$\begin{aligned} S_2(v_1 + \lambda v_2) &= (\alpha_1 + \lambda \beta_1) b_1 + (\alpha_2 + \lambda \beta_2) b_2 \\ &= \alpha_1 b_1 + \alpha_2 b_2 + \lambda (\beta_1 b_1 + \beta_2 b_2) \\ &= S_2(v_1) + \lambda S_2(v_2). \end{aligned}$$

S_0 preserves addition + scalar mult.

(2 marks A)

Idempotent:

Let $v \in V$ $v = \alpha_1 b_1 + \dots + \alpha_n b_n$.

$$S_2 \circ S_2(v) = S_2(S_2(v)) = S_2(\alpha_1 b_1 + \alpha_2 b_2) = \alpha_1 b_1 + \alpha_2 b_2 = S_2(v)$$

So S_2 is idempotent.

(1 mark C)

iv. $S_3 = (S_2)^2 = S_2 \circ S_2$

By above $S_2^2 = S_2$ which is both linear and idempotent

(1 mark B)

v. $S_4: V \rightarrow V$ $S_4(V) = \alpha_2 b_1 + \alpha_1 b_2$

Linear:

Let $v_1, v_2 \in V$, $\lambda \in \mathbb{R}$, with $v_1 = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$
 $v_2 = \beta_1 b_1 + \dots + \beta_n b_n$

$$\text{So } v_1 + \lambda v_2 = (\alpha_1 + \lambda \beta_1) b_1 + \dots + (\alpha_n + \lambda \beta_n) b_n.$$

$$\begin{aligned} S_4(v_1 + \lambda v_2) &= (\alpha_2 + \lambda \beta_2) b_1 + (\alpha_1 + \lambda \beta_1) b_2 \\ &= \alpha_2 b_1 + \alpha_1 b_2 + \lambda (\beta_2 b_1 + \beta_1 b_2) \\ &= S_4(v_1) + \lambda S_4(v_2). \end{aligned}$$

So preserves addition + scalar mult.

(2 marks A)

Idempotent:

Let $v = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$.

$$S_4 \circ S_4(v) = S_4(S_4(v)) = S_4(\alpha_2 b_1 + \alpha_1 b_2) = \alpha_1 b_1 + \alpha_2 b_2 \neq S_4(v)$$

So S_4 is NOT idempotent.

(1 mark C)

- (b) Show that $T : V \rightarrow V$ and $S : V \rightarrow V$ are linear transformations then $S \circ T$ is a linear transformation.

Let $v_1, v_2 \in V$ $\lambda \in \mathbb{R}$.

PRESERVES ADDITION:

$$\begin{aligned} S \circ T(v_1 + v_2) &= S(T(v_1 + v_2)) = S(T(v_1) + T(v_2)) && T \text{ linear} \\ &= S(T(v_1)) + S(T(v_2)) && S \text{ linear} \\ &= S \circ T(v_1) + S \circ T(v_2) \end{aligned}$$

(1 mark B)

PRESERVES SCALAR MULT:

$$\begin{aligned} S \circ T(\lambda v_1) &= S(T(\lambda v_1)) = S(\lambda T(v_1)) && T \text{ linear} \\ &= \lambda S(T(v_1)) && S \text{ linear} \\ &= \lambda(S \circ T(v_1)) \end{aligned}$$

(1 mark B)

- (c) Suppose $T : V \rightarrow V$ and $S : V \rightarrow V$ are idempotent is $S \circ T$ idempotent?

Sometimes $S \circ T$ is idempotent, e.g. choose any S idempotent, let $T = S$ then $S \circ T = S^2$ and

$$(S \circ S)(v) = S(S(v)) = S^2(v) \text{ so } S^2 \text{ is idempotent.}$$

(1 mark C)

Let $V = \mathbb{R}^2$ and for $v = \begin{pmatrix} a \\ b \end{pmatrix}$ define

$$T(v) = \begin{pmatrix} a+b \\ 0 \end{pmatrix} \quad S(v) = \begin{pmatrix} a \\ a \end{pmatrix}$$

$$\begin{aligned} \text{Then } T^2(v) &= T(T(v)) = T\left(\begin{pmatrix} a+b \\ 0 \end{pmatrix}\right) = \begin{pmatrix} a+b+0 \\ 0 \end{pmatrix} = T(v) \\ S^2(v) &= S(S(v)) = S\left(\begin{pmatrix} a \\ a \end{pmatrix}\right) = \begin{pmatrix} a \\ a \end{pmatrix} = S(v) \end{aligned}$$

So both S and T are idempotent.

$$S \circ T \begin{pmatrix} a \\ b \end{pmatrix} = S \begin{pmatrix} a+b \\ 0 \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \end{pmatrix}$$

$$(S \circ T)^2 \begin{pmatrix} a \\ b \end{pmatrix} = S \circ T \begin{pmatrix} a+b \\ a+b \end{pmatrix} = \begin{pmatrix} 2a+2b \\ 2a+2b \end{pmatrix} \neq S \circ T \begin{pmatrix} a \\ b \end{pmatrix}.$$

But $S \circ T$ is not idempotent.

So $S \circ T$ is not always idempotent if S and T are

(4 marks D).

- (d) When can you construct idempotent linear transformation $T : V \rightarrow V$ such that $\text{Im}(T) = \text{ker}(T)$.

By rank nullity theorem we have

$$\dim(V) = \dim(\text{Im}(T)) + \dim(\text{ker}(T))$$

So if $\text{Im } T = \text{ker } T$ we get $\dim(V) = 2 \dim(\text{Im}(T))$

So we at least need n to be even.

(1 mark C).

Now suppose $n = 2m$ for some $m \in \mathbb{N}$. Define

$$T(b_{2i+1}) = b_{2i+2} \quad \text{for } i = 0, \dots, m-1$$

$$T(b_{2i}) = 0_V \quad \text{for } i = 1, \dots, m.$$

Then $\text{Im}(T) = \text{Span}\{b_{2i}\}_{i=1, \dots, m} = \text{ker}(T)$,

So whenever V has even dimension we can define such a linear transformation.

(2 marks D)

Notion of idempotent unseen. (a) L.T part: seen similar.

(b) seen (c), (d) unseen.

4. (a) The sequences of real numbers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ satisfy the relations:

$$a_{n+1} = 3a_n + 2b_n \text{ and } b_{n+1} = -4a_n - 3b_n \text{ for all } n \geq 1.$$

- (i) Write down a matrix A with the property that

$$A \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}$$

and compute its eigenvalues and eigenvectors (over $\mathbb{R}$), showing the details of your computations.

(6 marks)

- (ii) Hence, or otherwise, give general expressions for a_n and b_n in terms of n, a_1 and b_1 .

(3 marks)

- (iii) Which values for a_1 and b_1 imply that both $a_{17} = 3$ and $b_{18} = 12$?

(1 mark)

- (b) (i) Suppose $v_1, \dots, v_r \in \mathbb{R}^n$ is an orthogonal set of non-zero vectors. Prove that for all $\alpha_1, \dots, \alpha_r \in \mathbb{R}$ we have

$$\left\| \sum_{i=1}^r \alpha_i v_i \right\| = \left(\sum_{i=1}^r \alpha_i^2 \|v_i\|^2 \right)^{\frac{1}{2}}.$$

(2 marks)

- (ii) Suppose $A \in M_n(\mathbb{R})$ is symmetric. Let α be the maximum value of $|\lambda|$ for all eigenvalues λ of A .

- Using (b)(i), show that $\|Av\| \leq \alpha\|v\|$ for all $v \in \mathbb{R}^n$ and there is some non-zero v for which we have equality. Clearly state any results from the notes which you require.
- Deduce that $\sup\{\|Av\| : \|v\| = 1\} = \alpha$.
- State and prove a similar result for $\inf\{\|Av\| : \|v\| = 1\}$.

(8 marks)

4. *Solution:*

(a) (i) We take $A = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$.

1A

The characteristic polynomial is $\det(xI_2 - A) = (x - 3)(x + 3) + 8 = (x - 1)(x + 1)$, so the eigenvalues are 1 and -1 .

Solving $(I_2 - A)v = 0$ gives that the eigenspace E_1 is spanned by $(1, -1)^T$. Similarly the eigenspace E_{-1} is spanned by $(-1, 2)^T$.

4A

The eigenvectors are the non-zero vectors in these.

1A

- (ii) We observe that $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = A^{n-1} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$. If $P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ then $P^{-1}AP$ is the diagonal matrix $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so $A^k = PD^kP^{-1}$, and this can easily be computed by computing $P^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and simplifying.

However, we can also note that $A^2 = I_2$ (for example, as $D^2 = I_2$), so $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = A^{n-1} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ if n is odd, and $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = A \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 3a_1+2b_1 \\ -4a_1-3b_1 \end{pmatrix}$ if n is even. **3B**

- (iii) If $a_{17} = 3$, then by (iii) $a_1 = 3$; so if also $b_{18} = 12$, then $b_1 = -8$. **1A**

(b) (i)

$$\left\| \sum_{i=1}^r \alpha_i v_i \right\|^2 = \left(\sum_{i=1}^r \alpha_i v_i \right) \cdot \left(\sum_{j=1}^r \alpha_j v_j \right) = \sum_{i,j=1}^r \alpha_i \alpha_j v_i \cdot v_j = \sum_i \alpha_i^2 \|v_i\|^2$$

as $v_i \cdot v_j = 0$ if $i \neq j$. Taking non-negative square roots gives the result. **2B**

- (ii) As A is symmetric, there is an orthonormal basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ consisting of eigenvectors of A . Let $Av_i = \lambda_i v_i$. If $v \in \mathbb{R}^n$ let $v = \sum_{i=1}^n \alpha_i v_i$. Then $Av = \sum_{i=1}^n \alpha_i \lambda_i v_i$, so by (i) and orthonormality of the v_i we have

$$\|v\| = \left(\sum_i \alpha_i^2 \right)^{1/2} \text{ and } \|Av\| = \left(\sum_i \alpha_i^2 \lambda_i^2 \right)^{1/2}.$$

As $0 \leq \lambda_i^2 \leq \alpha^2$ for all i , we obtain the result. **4D**

By the above, the supremum is at most α . Suppose $|\lambda_i| = \alpha$. We have $\|Av_i\| = \alpha$ and so the supremum is precisely α . **1A**

If β is the minimum value of $|\lambda_j|$, then $\inf(\|Av\| : \|v\| = 1) = \beta$. We have $\lambda_j^2 \geq \beta^2 \geq 0$ for all j and so the above calculation gives $\|Av\| \geq \beta\|v\|$ for all $v \in \mathbb{R}^n$. So the infimum here is at least β . Taking v to be v_j where $|\lambda_j| = \beta$ then gives equality. **3C**

[Comments: (a) did examples like this in the notes; the $A^2 = I_2$ observation is unseen. (b) Unseen.]

5. (a) Let $n \geq 2$ and let F be a field. Suppose $A \in M_n(F)$ has exactly n non-zero entries $x_1, \dots, x_n$, one in each row and one in each column of A .

- (i) Prove that $\det(A) = \pm x_1 x_2 \dots x_n$.
(ii) One of the zero entries in A is changed to 1, resulting in the matrix B . Express $\det(B)$ in terms of $\det(A)$, explaining your answer.

(6 marks)

- (b) Suppose $C \in M_4(F)$ and $\lambda \in F$ is an eigenvalue of C such that the eigenspace E_λ has dimension 2. Prove that $\chi_C(x) = (x - \lambda)^2 g(x)$ for some polynomial $g(x)$ with coefficients in F . (You may quote results from the notes, but you should not quote general results which immediately imply what is to be proved.) **(6 marks)**

- (c) Let G be the symmetric group S_6 and $H = \{h \in G : \{h(1), h(2)\} = \{1, 2\}\}$.

- (i) Show that H is a subgroup of G and determine $|H|$, explaining your answer. (3 marks)

- (ii) Prove that for $g, k \in G$ we have

$$gH = kH \Leftrightarrow \{g(1), g(2)\} = \{k(1), k(2)\}.$$

(3 marks)

- (iii) Using (c)(i) and (ii), compute the index of H in G in two different ways. (2 marks)

(Throughout the question, results from the notes may be used if clearly stated.)

5. Solution:

- (a) (i) By interchanging the rows of A we can obtain a matrix D where the non-zero entries are all on the diagonal. This matrix has determinant $\pm \det(A)$. We take factors $x_1, \dots, x_n$ (in some order) from the rows of D and obtain the identity matrix, which has determinant 1. Thus $\det(A) = \pm x_1 \dots x_n$. **3A**
- (ii) Suppose the entry of A which is changed is the ij -th. There is some row of A , say row k , with a non-zero entry in column j . Note that $k \neq i$ as $a_{ij} = 0$. By subtracting a multiple of row k from row i in the new matrix, we obtain A . Thus $\det(B) = \det(A)$. [Remark: There may be some solutions which use the formula for the determinant in terms of permutations: these can be allowed if explained carefully.] **3A**
- (b) Let v_1, v_2 be a basis for E_λ and extend this to a basis $v_1, \dots, v_4$ for F^4 . The matrix representing the map $T_C : F^4 \rightarrow F^4$ given by $v \mapsto Cv$ with respect to this basis is of the form

$$\begin{pmatrix} \lambda & 0 & * \\ 0 & \lambda & * \\ 0 & 0 & A \end{pmatrix}$$

where $A \in M_2(F)$. The characteristic polynomial of C is equal to the characteristic polynomial of this and is the determinant of

$$\begin{pmatrix} x - \lambda & 0 & * \\ 0 & x - \lambda & * \\ 0 & 0 & xI_2 - A \end{pmatrix}.$$

Expanding down column 1, and then down column 2 in the 11-minor gives that this polynomial is $(x - \lambda)^2 \chi_A(x)$, hence the result. **6C/D**

- (c) (i) Use the subgroup test (noting that as G is a finite group, we do not have to check closure under inverses). First, note that $H \neq \emptyset$ as the identity element is in H . Suppose $h, k \in H$. Then $\{(hk)(1), (hk)(2)\} = \{h(1), h(2)\}$ (in some order) and this is $\{1, 2\}$. So H is closed under the group operation. The elements of H consist of a permutation of $\{1, 2\}$ and a permutation of $\{3, 4, 5, 6\}$ and there are $2 \cdot (4!) = 48$ of these. **3A**

- (ii) If $gH = kH$ then $k = gh$ for some $h \in H$ and thus $\{k(1), k(2)\} = \{gh(1), gh(2)\} = \{g(1), g(2)\}$. Conversely, if $\{k(1), k(2)\} = \{g(1), g(2)\}$, then $k^{-1}g(1), k^{-1}g(2) \in \{1, 2\}$. It follows that $\{k^{-1}g(1), k^{-1}g(2)\} = \{1, 2\}$, so $k^{-1}g \in H$. Thus, $gH = kH$.

3B

- (iii) As $|G| = 6!$ and $|H| = 2!4!$ (part (i)), the index of H in G is $6!/2!4! = 15$. Using part (ii), the number of cosets of H in G is the number of possibilities for the set $\{g(1), g(2)\}$ (for $g \in G$). This can be any subset of $\{1, 2, 3, 4, 5, 6\}$ of size 2, so is $\binom{6}{2} = 15$. **2B**
[Remark: we did not cover actions and orbit-stabilizer in the lectures.]

[Comment: Almost all unseen; (c)(i) was on a problem sheet.]

6. (a) Suppose $(G, \cdot)$ is a group and let $H = \{g^2 : g \in G\}$. Three of the following four statements are true and one is not true in general. Prove the ones which are true and give, with explanation, a counterexample for the other one.
- (i) If G is abelian, then H is a subgroup of G .
 - (ii) If G is finite and $|G|$ is odd, then $H = G$.
 - (iii) If $n \geq 4$ and G is the symmetric group S_n , then $H = G$.
 - (iv) If G is the dihedral group D_{2n} , then H is a subgroup of G .

(8 marks)

- (b) Consider the following elements of S_9 :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 & 1 \end{pmatrix}; g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

- (i) Write $f, g, f^{-1}, g^3 f g f^2$ in disjoint cycle form and state the orders of these. (4 marks)
- (ii) What are the cycle shapes of elements of order 3 in S_9 ? How many elements of order 3 are there in S_9 ? (You may leave your answers expressed in terms of binomial coefficients.)

(4 marks)

- (c) Suppose G is a finite group. Prove that the number of elements of order 5 in G is divisible by 4. State and prove a more general result about the number of elements of order p in G , where p is a prime number.

(4 marks)

(Throughout the question, results from the notes may be used if clearly stated.)

6. *Solution:*

- (a) In the following, no marks for correct T/F: the marks are for the reasoning.

TRUE: Use the subgroup test. First, $e \in H$ so $H \neq \emptyset$. If $g, h \in H$, then as G is abelian $(gh)^2 = ghgh = gghh = e$, so H is closed under the operation. Finally $(g^{-1})^2 = (g^2)^{-1} = e$, so H is closed under inverses. **2A**

- (ii) TRUE: Suppose $g \in G$. This has odd order (by a corollary to Lagrange's theorem), say $2k - 1$ for some $k \in \mathbb{N}$. Thus $g^{2k-1} = e$ and so $g = g^{2k} = (g^k)^2 \in H$. **2C**

(iii) FALSE: It's not true for $n = 4$. We can see this directly by computing the possible cycle shapes and the squares of these: we observe that no square is equal to (12) or (1234) . But we can also see this from the sign homomorphism: the sign of g^2 is $+1$, so no odd permutation is in H . **2C**

(iv) TRUE: The elements of D_{2n} are either rotations in the plane of the regular n -gon, or reflections. The square of a reflection is just the identity. So H consists of the squares of the rotations. The rotations form an abelian group, so H is a subgroup by (i). **2B**

(b) (i) $f = (13579)(2468)$ and $g = (19)(28)(37)(46)$. So $f^{-1} = (97531)(8642)$. As $g^3 = g$ we compute:

$$g^3 f g f^2 = (19)(28)(37)(46)(13579)(2468)(19)(28)(37)(46)(15937)(26)(48) = (13579)(2468).$$

The order of a permutation in disjoint cycle form is the l.c.m. of the lengths of its cycles so the orders of these elements are, respectively, 20, 2, 20, 20. **4A**

(ii) The possible cycle shapes here are $3^1, 1^6$ (that is, one 3-cycle and 6 fixed points), $3^2, 1^3$ and 3^3 . **1A**

The number of elements of each of these types is:

Shape $3^1, 1^6$: $2 \cdot \binom{9}{3}$ (choose the set for the support of the 3-cycle; each set gives 2 possible 3-cycles);

Shape $3^2, 1^3$: $2 \binom{9}{3} \cdot 2 \binom{6}{3} / 2$ (choose the first 3-cycle as above; then choose the second one from the remaining 6 points; each permutation gets counted twice).

Shape 3^3 : $2^3 \binom{9}{3} \binom{6}{3} / 6$ (similar to above, but now each permutation is counted $3!$ times.)

[Note: some students will have found a formula; I suppose we have to accept this.] **3B**

(c) Each element of order 5 is contained in exactly one subgroup of order 5, and each of the four non-identity elements in the subgroup generates the whole subgroup (eg by Lagrange's theorem). It follows that the number of elements of G of order 5 is 4 times the number of subgroups of G of order 5. In particular, 4 divides the number of elements of order 5 in G . The same argument works with an arbitrary prime p in place of 5 and we obtain that $p - 1$ divides the number of elements of order p in G . **4D**

[Comment: (a) is unseen; did lots of examples like (b)(i), (ii); (c) is unseen.]

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH40003	1	1(a) I was disappointed with how many of the students found row reduction difficult – especially given the online tools available to them. Those that did not make calculation errors did well on this part. 1(b) Many students did not do this from first principles – forgetting to explain (for example) why $0_V = 0_W$. 1(c) Several students did not answer this question fully – many forgetting particular values for alpha. Those that answered all parts tended to do so quite well, though many missed that 1(c)(ii) is a subspace for all alpha. There was a typo in 1(c)(iii) the vector should have read $(-13/9, 1/9, 0)$, I was lenient with the marking for this.
MATH40003	2	In question 2(c)-ii there was some confusion about the matrix. In question 2(d) – i the space V^* is not a vector – it is the set of all linear transformations. Linearity of T^* needs to be properly checked for the relevant space.
MATH40003	3	By the given definition idempotent transformations are linear. So if we show a transformation is not linear automatically it is not idempotent. This also caused problems in some cases for 3(c): many provided examples of maps that were not LT. When the question asks about the full justification please provide a complete justification.

MATH40003	4	(a)(i). A non-zero eigenvalue can't have the zero vector as an eigenvector, so when parametrising the eigenvector here, you had to specify that the parameter was non-zero. (a)(ii). You had to actually calculate the final formulas, anything where I had to multiply matrices out to finish the calculation was regarded as incomplete. Very few people spotted that since the eigenvalues were the square roots of 1, that meant that the square of A was the identity. Your hint to that was when you wrote out the diagonal matrix. It was exceedingly common to calculate A^n, and then use that to find (a_n, b_n), but you would need A^{n-1} there. Many people would have benefited from checking if their formulas worked for $n=1$. a(iii). There were a lot of basic mistakes in this question. Using 17 for the value of a_{17}, instead of 3, or getting a minus sign confused, were the most common. b(i). This result is Pythagoras' theorem, so you had to actually prove it, not just quote it. I also needed some argument about the fact that orthogonality meant that the dot product of different basis vectors was zero. b(ii). A was fixed in this question. It started with "Suppose A...", so it was quite clear that you needed to let v vary to find when the inequality was an equality, not A. You couldn't simply use the same v's from (i) here, you needed to use your notes to show that there is an orthonormal basis of eigenvectors of A.
MATH40003	5	Parts a and c were generally solved well. There were quite a few people assuming that the subgroup H was fixing 1 and 2, but this was not in the text. In Part b very few people produced a proof that did not use results that implied it directly. Saying that since we have an eigenspace of dimension 2, then there must be no more than 3 eigenvalues, or equivalent arguments, were concluding directly from the contrapositive of a result in the notes, so they were not given full marks.
MATH40003	6	Part (a) was generally solved well. No marks were given for correct True/False, but just for proves and counter-examples. Part (b) was also done well. Those that did not justify the computations in b (ii) did not receive full marks, despite writing the correct answer. In Part (c) many forgot to show that two distinct cyclic subgroups of order 5 intersected only trivially.