

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Linear Algebra and Groups

Date: 10 May 2022

Time: 09:00 – 12:00 (BST)

Time Allowed: 3:00 hours

Upload Time Allowed: 45 minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Consider the following system of linear equations:

$$x + y + 70z = 16$$

$$0 + y + 35z = 8$$

$$x + y + 105z = 23$$

Either find all the solutions, or show that no solutions exist when:

- (i) $x, y, z \in \mathbb{R}$. (2 marks)
 - (ii) $x, y, z \in \mathbb{Z}_5$. (3 marks)
 - (iii) $x, y, z \in \mathbb{Z}_7$. (3 marks)
- (b) Let V be a vector space over field F , and let $u, v, w \in V$. Prove from first principles, i.e. using only the vector space axioms, that if $v + u = w + v$ then $u = w$. (4 marks)
- (c) Let V and W be vector spaces over the same field F . We put the following operations on the set $V \times W$:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$

$$\lambda(v_1, w_1) = (\lambda v_1, \lambda w_1)$$

- (i) Prove that $V \times W$ with these operations satisfies the vector space axioms A1, A3. (4 marks)
- (ii) You may now assume that $V \times W$ is a vector space. Prove that if $A \leq V$ and $B \leq W$ then $A \times B \leq V \times W$. (4 marks)

(Total: 20 marks)

2. Let $\mathbb{M}$ be the set of all 2×2 matrices with entries in $\mathbb{R}$, and let $\lambda \in \mathbb{R}$. You may assume that $\mathbb{M}$, with the operations

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

$$\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$$

is a vector space.

Let B be the standard basis for $\mathbb{M}$. i.e.

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

- (a) Prove that

$$B' = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

is a basis of $\mathbb{M}$.

(4 marks)

- (b) Let $T : \mathbb{M} \rightarrow \mathbb{M}$ be given by $T(A) = A^T$.

Calculate the following matrices:

- (i) ${}_B[T]_B$ (2 marks)
- (ii) ${}_{B'}[T]_{B'}$ (2 marks)
- (iii) ${}_B[id]_{B'}$ (3 marks)

- (c) Prove that if $S : \mathbb{M} \rightarrow \mathbb{M}$ is a linear transformation such that there is a basis D of $\mathbb{M}$ where ${}_D[T]_D$ is the identity matrix, then S is the identity map. (7 marks)

- (d) Let $S : \mathbb{M} \rightarrow \mathbb{R}$ be the linear transformation given by $S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + b$. Calculate the rank and nullity of S . (2 marks)

(Total: 20 marks)

3. Let V be a vector space. A linear transformation $T : V \rightarrow V$ is said to be *nilpotent* if there is an $m \in \mathbb{N}$ such that $T^m(v) = 0$ for all $v \in V$.

- (a) Suppose V has basis $\{b_1, \dots, b_n\}$. Let T be the following function:

$$T\left(\sum_{i=1}^n \alpha_i b_i\right) = \sum_{i=1}^{n-1} \alpha_{i+1} b_i$$

Prove that T is a nilpotent linear transformation. (2 marks)

- (b) Let T be any linear transformation of V , and let

$$\text{Nil}(T) = \{v \in V : \exists m \in \mathbb{N}, T^m(v) = 0\}$$

Prove that $\text{Nil}(T)$ is a subspace of V . (4 marks)

- (c) Let T be a nilpotent linear transformation from V to itself. The *degree* of T is the smallest m such that $T^m(v) = 0$ for all v . Prove that the degree of T is less than or equal to the dimension of V . (8 marks)

- (d) Suppose that T is nilpotent. Prove that $I - T$ and $I + T$ are invertible. (6 marks)

(Total: 20 marks)

4. (a) Let V be a 3-dimensional vector space over $\mathbb{R}$ and let $B = \{v_1, v_2, v_3\}$ be a basis of V . Let $T : V \rightarrow V$ be the linear transformation defined by

$$T(v_1) = -2v_3, \quad T(v_2) = 2v_2 - 3v_3, \quad T(v_3) = -v_1 + v_3.$$

- (i) Write down the matrix $[T]_B$, explaining your answer. (2 marks)
- (ii) Find the eigenspaces of T (show the details of your calculation), and determine whether T is diagonalisable or not (explain your answer). (6 marks)
- (b) For $x, y, z \in \mathbb{R}$, let $A_{x,y,z} \in M_4(\mathbb{R})$ be the following matrix:

$$A_{x,y,z} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & x+2 & 1 & 1 \\ 1 & 1 & y+1 & 1 \\ 1 & 2 & 1 & z+1 \end{pmatrix}.$$

- (i) Determine all the triples $(x, y, z) \in \mathbb{R}^3$ for which $A_{x,y,z}$ is invertible. (3 marks)
- (ii) Suppose $yz \neq 0$ and $B \in M_4(\mathbb{R})$ is such that $\det(B) = xyz$. Is there an invertible matrix $P \in M_4(\mathbb{R})$ such that $P^{-1}BP = A_{x,y,z}$? Explain your answer. (2 marks)
- (c) Let $n \in \mathbb{N} \setminus \{0\}$. Let $A \in M_n(\mathbb{R})$ be invertible. Show that there are $B, C \in M_n(\mathbb{R})$ such that B is orthogonal, C is upper-triangular, and

$$A = BC.$$

(7 marks)

(Total: 20 marks)

5. (a) (i) Check if the following vectors form an orthogonal basis of $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix}.$$

(3 marks)

- (ii) Let $n \in \mathbb{N} \setminus \{0\}$. Let E be the standard basis of $\mathbb{R}^n$. Assume that $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation such that, for all $u, v \in \mathbb{R}^n$,

$$T(u) \cdot v = u \cdot T(v).$$

Prove that $[T]_E$ is symmetric.

(3 marks)

- (iii) Let $n \in \mathbb{N} \setminus \{0\}$ and let A be an $n \times n$ real symmetric matrix. Show that the characteristic polynomial of A splits as a product of (possibly repeated) factors of degree 1. (You may quote results from the notes in your proof.)

(2 marks)

- (b) Let $n \in \mathbb{N} \setminus \{0\}$. Recall that

$$O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid AA^T = I_n\}.$$

Show that if T is a rotation of $\mathbb{R}^2$ and E is the standard basis of $\mathbb{R}^2$, then $[T]_E \in O_2(\mathbb{R})$.

(3 marks)

- (c) Let $n \in \mathbb{N} \setminus \{0\}$ and let F be a field. Recall that $\text{GL}_n(F)$ is the group of all $n \times n$ invertible matrices. Let H be the set of matrices in $M_n(F)$ having exactly one entry equal to 1 for each row and column and all other entries equal to 0.

- (i) Show that, if $F = \mathbb{R}$, then $H \subseteq O_n(\mathbb{R})$. (2 marks)
- (ii) Show that H is a subgroup of $\text{GL}_n(F)$ and that H is isomorphic to the symmetric group S_n . (7 marks)

(Total: 20 marks)

6. (a) Let G be the group $(\mathbb{R}^2, +)$, where $+$ is the usual vector addition. Let

$$H = \{(x, 2x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}.$$

- (i) Show that H is a subgroup of G . (2 marks)
- (iii) Show that H is isomorphic to $(\mathbb{R}, +)$, where $+$ is the usual addition in $\mathbb{R}$. (2 marks)
- (iv) Let A be the set of all left cosets of H in G . Show that there is a binary operation $*$ on A such that $(A, *)$ is a group isomorphic to $(\mathbb{R}, +)$. (4 marks)

(b) Let σ be the permutation in S_9 given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 9 & 6 & 3 & 4 & 1 & 5 & 2 & 8 \end{pmatrix}.$$

Give the decomposition of σ into disjoint cycles. Justify your answer. (3 marks)

(c) Let G be a group and H be a subgroup of G such that $|G : H| = 2$.

- (i) Show that $\{gH \mid g \in G\} = \{Hg \mid g \in G\}$. (2 marks)
- (ii) Let now

$$H = \{\sigma \in S_5 \mid \text{sgn}(\sigma) = 1\}.$$

Show that $\{gH \mid g \in S_5\} = \{Hg \mid g \in S_5\}$. (3 marks)

(d) Let G be a finite group. Show that there is $n \in \mathbb{N}$ such that G is isomorphic to a subgroup of S_n . (4 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2022

This paper is also taken for the relevant examination for the Associateship.

MATH40003

Linear Algebra and Groups (Solutions)

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1. (a) If we create an augmented matrix for the system of linear equations and reduce it to row-reduced echelon form, we get

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seen ↓

$$\begin{pmatrix} 1 & 1 & 70 & | & 16 \\ 0 & 1 & 35 & | & 8 \\ 1 & 1 & 105 & | & 23 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 70 & | & 16 \\ 0 & 1 & 35 & | & 8 \\ 0 & 0 & 35 & | & 7 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 35 & | & 7 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 35 & | & 7 \end{pmatrix}$$

This last matrix is the best we can get without dividing by either 5 or 7.

- (i) If the field is $\mathbb{R}$ then the RREF matrix simplifies further, to

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & \frac{1}{5} \end{pmatrix}$$

and we have that $x = 1$, $y = 1$, and $z = \frac{1}{5}$.

2, A

- (ii) If the field is $\mathbb{Z}_5$ then the RREF matrix is

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & | & 2 \end{pmatrix}$$

this is inconsistent, and therefore there are no solutions.

3, B

- (iii) If the field is $\mathbb{Z}_7$ then the RREF matrix is

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

and we get that for all $\lambda \in \mathbb{Z}_7$ we have that $(x, y, z) = (1, 1, \lambda)$ is a solution.

3, B

- (b) If $v + u = w + v$ then applying A2 to the LHS gives us that $u + v = w + v$.

By A4 there is a $-v$, which we add to both sides to get $(u + v) + (-v) = (w + v) + (-v)$.

A1 lets us rearrange the brackets: $u + (v + (-v)) = w + (v + (-v))$.

A4 again: $u + 0 = w + 0$.

Finally, A3 gives $u = w$.

seen or
sim.seen
↓

4, A

- (c) (i) Addition and scalar multiplication have the correct domain and co-domain, therefore we just have to check the vector space axioms.

A1: Let $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in V \times W$:

$$\begin{aligned} ((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) \\ &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) \\ &= (v_1, w_1) + ((v_2, w_2) + (v_3, w_3)) \end{aligned}$$

2, A

A3: Let $(v_1, w_1) \in V \times W$:

$$\begin{aligned} (v_1, w_1) + (0_V, 0_W) &= (v_1 + 0_V, w_1 + 0_W) \\ &= (v_1, w_1) \end{aligned}$$

2, A

- (ii) Clearly, since both A and B are non-empty, we have that $A \times B \neq \emptyset$. It is also clear that $A \times B \subseteq V \times W$.

Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$. Since $A \leq V$, we have $a_1 + a_2 \in A$, and similarly $b_1 + b_2 \in B$, therefore $(a_1, b_1) + (a_2, b_2) \in A \times B$.

seen or
sim.seen
↓

2, B

1, A

Let $(a_1, b_1) \in A \times B$ and let λ be a scalar. Then $\lambda(a_1, b_1) = (\lambda a_1, \lambda b_1)$. Since $A \leq V$, we have $\lambda a_1 \in A$, and similarly $\lambda b_1 \in B$, therefore $\lambda(a_1, b_1) \in A \times B$.

1, A

2. (a)

$$b_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, b_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The following equations show that B is in the span of B' :

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= \frac{1}{3}(2b_1 - b_2) \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \frac{1}{3}(-b_1 + 2b_2) \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= b_3 - \left(\frac{1}{3}(2b_1 - b_2)\right) - \left(\frac{1}{3}(-b_1 + 2b_2)\right) \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= b_4 - \left(\frac{1}{3}(2b_1 - b_2)\right) - \left(\frac{1}{3}(-b_1 + 2b_2)\right) \end{aligned}$$

Please award full marks if they do this but use shortcuts like using $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ directly if they have already shown how it is in the span of B' . Since $|B| = |B'|$, B is a basis, B' is spanning, and V is finite dimensional, we know that B' is a basis.

(b) (i)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(iii) Using the equations from Part (a), we get

$${}_{B'}[\text{id}]_B = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

Inverting this matrix gives us

$${}_B[\text{id}]_{B'} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

(c) Suppose there is a basis D of $\mathbb{M}$ such that ${}_D[S]_D$ is the identity matrix, then let $D = \{d_1, \dots, d_4\}$. For all i , we have that $S(d_i) = d_i$. Since D is a basis, for every $M \in \mathbb{M}$ there are $\alpha_1, \dots, \alpha_4$ such that $M = \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 + \alpha_4 d_4$.

S is a linear transformation, so

$$\begin{aligned} S(M) &= S(\alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 + \alpha_4 d_4) \\ &= \alpha_1 S(d_1) + \alpha_2 S(d_2) + \alpha_3 S(d_3) + \alpha_4 S(d_4) \\ &= \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 + \alpha_4 d_4 \\ &= M \end{aligned}$$

Therefore, if there is such a D exists, then $S(M) = M$ for all M .

(d) Let $\lambda \in \mathbb{R}$. Then $S(\frac{\lambda}{2}\text{Id}) = \lambda$, so the image of S is all of $\mathbb{R}$, and therefore the rank is 1. By the Rank-Nullity Theorem the nullity of S must be 3.

3. (a) IF we apply the given formula repeatedly, we get

$$T^k\left(\sum_{i=1}^n \alpha_i b_i\right) = \sum_{i=1}^{n-k} \alpha_{i+k} b_i$$

Taking $k = n$, we get that $T^n(v) = 0$ for all v .

- (b) Clearly by definition, $\text{Nil}(T)$ is a subset of V .

Since T is a linear transformation, $T(0) = 0$, and therefore $\text{Nil}(T)$ is non-empty.

Let $u, v \in \text{Nil}(T)$, i.e. there are $n_u, n_v \in \mathbb{N}$ such that $T^{n_u}(u) = 0 = T^{n_v}(v)$. Then

$$\begin{aligned} T^{n_u+n_v}(u+v) &= T^{n_u+n_v}(u) + T^{n_u+n_v}(v) \\ &= T^{n_v}(T^{n_u}(u)) + T^{n_u}(T^{n_v}(v)) \\ &= T^{n_v}(0) + T^{n_u}(0) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore T preserves addition.

Let $v \in \text{Nil}(T)$, i.e. there is $n \in \mathbb{N}$ such that $T^n(v) = 0$, and let λ be a scalar. Then

$$\begin{aligned} T^n(\lambda v) &= \lambda T^n(v) \\ &= \lambda 0 \\ &= 0 \end{aligned}$$

Therefore T preserves scalar addition.

Therefore $\text{Nil}(T)$ is a subspace of V .

- (c) Suppose T is nilpotent, but that the degree of T (denoted by m) is strictly greater than $\dim(V)$, i.e. there is a $v \in V$ such that $T^{m-1}(v) \neq 0$. In particular, this means that $T^i(v) \neq 0$ for all $i < m$.

Let's investigate the set

$$X = \{v, T(v), T^2(v), \dots, T^{m-1}(v)\}$$

Suppose that this set is not linear independent, i.e. there are α_i such that

$$\sum_{i=0}^{m-1} \alpha_i T^i(v) = 0$$

using the convention that $T^0 = id$. If we apply T^{m-1} to both sides of this equation, we get that

$$\begin{aligned} T^{m-1}\left(\sum_{i=0}^{m-1} \alpha_i T^i(v)\right) &= T^{m-1}(0) \\ T^{m-1}(v) + \sum_{i=1}^{m-1} \alpha_i T^{m-1+i}(v) &= 0 \\ T^{m-1}(v) &= 0 \end{aligned}$$

This contradicts our assumptions about v and the degree of T . Therefore the set X is linearly independent. However $|X| = m$, which is greater than the dimension of V , so we have a second contradiction.

Therefore the degree of T is less than or equal to the dimension of V .

- (d) Let n be the degree of T .

$$\begin{aligned} (I - T) \sum_{i=0}^{n-1} T^i &= \sum_{i=0}^n (T^i - T^{i+1}) \\ &= I \end{aligned}$$

Similarly, the inverse of $(I + T)$ is $\sum_{i=0}^{n-1} (-1)^i T^i$

- (4) (a) (i) The images of the vectors in B are already given in terms of the basis B . ($T(v_1) = -2v_3, T(v_2) = 2v_2 - 3v_3, T(v_3) = -v_1 + v_3$.) Thus

$$A = [T]_B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -2 & -3 & 1 \end{pmatrix}.$$

(2 marks) **Cat A** Seen

- (ii) The characteristic polynomial of A is $\det(XI_3 - A) = (X + 1)(X - 2)^2$. Thus there are two eigenvalues: -1 and 2 .

(2 marks) **Cat A** Seen

The eigenspace for -1 is

$$E_{-1} = \ker(-I_3 - A) = \ker \begin{pmatrix} -1 & 0 & 1 \\ 0 & -3 & 0 \\ 2 & 3 & -2 \end{pmatrix} = \text{Span} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

We are working with coordinates in the basis B so $E_{-1} = \text{Span}(v_1 + v_3)$.

The eigenspace for 2 is

$$E_2 = \ker \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 3 & 1 \end{pmatrix} = \text{Span} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

We are working with coordinates in the basis B so $E_2 = \text{Span}(-v_1 + 2v_3)$

(2 marks) **Cat A** Seen

The eigenvalue 2 has multiplicity 2 as a root of χ_T but E_2 has dimension 1 . It follows that it is not possible to find a basis of eigenvectors for T and therefore T is not diagonalisable.

(2 marks) **Cat A** Seen

- (b) (i) The matrix $A_{x,y,z}$ is invertible if and only if $\det(A_{x,y,z}) \neq 0$. We compute $\det(A_{x,y,z})$ as a function of x, y, z : subtracting the first row to all the others we see that $A_{x,y,z}$ is row-equivalent to

$$A' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & x+1 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 1 & 0 & z \end{pmatrix}.$$

A Laplace expansion along the first column of A' gives that

$$\det(A_{x,y,z}) = \begin{vmatrix} x+1 & 0 & 0 \\ 0 & y & 0 \\ 1 & 0 & z \end{vmatrix} = yz(x+1).$$

We conclude that $A_{x,y,z}$ is invertible for all triples in

$$\{(x, y, z) \in \mathbb{R}^3 \mid x \neq -1, y \neq 0, z \neq 0\}.$$

(3 marks) **Cat B** Unseen but standard

- (ii) The answer is no. Suppose a P as in the question exists, then by the previous part and the fact that $P^{-1}BP$ has determinant equal to $\det(B)$, we deduce

$$yz(x+1) = xyz.$$

Since $yz \neq 0$, this implies that $x+1 = x$, that is $1 = 0$, which is a contradiction.

(2 marks) **Cat B** Unseen but standard

- (c) (Note: A is assumed invertible only Gram-Schmidt is stated for linearly independent vectors in the notes). Let $v_1, \dots, v_n$ be the columns of A . We apply the Gram-Schmidt process to

$v_1, \dots, v_n$ to obtain an orthogonal basis $b_1, \dots, b_n$ from the columns of A .

$$\begin{aligned}
 b_1 &= v_1; & v_1 &= b_1; \\
 b_2 &= v_2 - \frac{b_1 \cdot v_2}{b_1 \cdot b_1} b_1; & v_2 &= b_2 + \frac{b_1 \cdot v_2}{b_1 \cdot b_1} b_1; \\
 b_3 &= v_3 - \left(\frac{b_1 \cdot v_3}{b_1 \cdot b_1} b_1 + \frac{b_2 \cdot v_3}{b_2 \cdot b_2} b_2 \right); & v_3 &= b_3 + \left(\frac{b_1 \cdot v_3}{b_1 \cdot b_1} b_1 + \frac{b_2 \cdot v_3}{b_2 \cdot b_2} b_2 \right); \\
 \vdots & & \vdots & \\
 b_i &= v_i - \sum_{j=1}^{i-1} \left(\frac{b_j \cdot v_i}{b_j \cdot b_j} \right) b_j; & v_i &= b_i + \sum_{j=1}^{i-1} \left(\frac{b_j \cdot v_i}{b_j \cdot b_j} \right) b_j; \\
 \vdots & & \vdots &
 \end{aligned}$$

(3 marks) **Cat C** Unseen

For $i = 1, \dots, n$, the above gives

$$v_i \in \text{Span}_{\mathbb{R}}(b_1, \dots, b_i).$$

(2 marks) **Cat D** Unseen

Let B be the matrix formed by juxtaposing the column vectors $u_1 = b_1/\|b_1\|, \dots, u_n = b_n/\|b_n\|$. Clearly, by the above,

$$(1) \quad v_i \in \text{Span}_{\mathbb{R}}(u_1, \dots, u_i).$$

Now, if $M = (m_{ij}) \in M_n(\mathbb{R})$, then the j th column of BM is $\sum_{i=1}^n m_{ij} b_i$. For $i, j = 1, \dots, n$, let $c_{ij} \in \mathbb{R}$ be such that

$$v_j = \sum_{i=1}^n c_{ij} u_i.$$

Then $C = (c_{ij})$ is upper triangular by (1) and $BC = A$.

(2 marks) **Cat D** Unseen

- (5) (a) (i) We check that $v_1 \cdot v_2 = -2 + 2 = 0$, $v_1 \cdot v_3 = 6 - 6 = 0$, and $v_2 \cdot v_3 = -6 - 12 + 15 = 0$. Hence $\{v_1, v_2, v_3\}$ is an orthogonal basis of $\mathbb{R}^3$. (3 marks) **Cat A Seen**
 (ii) Let $A = [T]_E$. Then, for $u, v \in \mathbb{R}^n$, $T(u) \cdot v = Au \cdot v$ and $u \cdot T(v) = u \cdot Av$. Hence $T(u) \cdot v = u \cdot T(v) = u \cdot Av$ if and only if $Au \cdot v = u \cdot Av$. We rewrite this as last equation as

$$u^T A^T v = u^T Av.$$

This holds for arbitrary u, v (thus in particular for u, v in a basis of $\mathbb{R}^4$) and hence it implies $A = A^T$. (3 marks) **Cat A Seen**

- (iii) By the Spectral Theorem, A is diagonalisable in $\mathbb{R}$. Let $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ be a diagonal matrix such that $P^{-1}AP = D$ for some invertible $n \times n$ real matrix P . Then $\chi_A(X) = \prod_{i=1}^n (X - \lambda_i)$ as required. (2 marks) **Cat A Seen**

- (b) Let $E = \{e_1, e_2\}$. A rotation of $\mathbb{R}^2$ is such that $T(e_1) = (\cos \theta)e_1 + (\sin \theta)e_2$ and $T(e_2) = -(\sin \theta)e_1 + (\cos \theta)e_2$ for some $\theta \in [0, 2\pi)$. We deduce that

$$A = [T]_E = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

(2 marks) **Cat B Seen**

Therefore

$$AA^T = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = I_2,$$

proving that $A \in O_n(\mathbb{R})$. (1 mark) **Cat B Seen**

- (c) (i) Let $A = (a_{ij})$ have columns $c_1, \dots, c_n$. Then for all $k, \ell = 1, \dots, n$,

$$c_\ell \cdot c_k = \sum_{i=1}^n a_{i\ell} a_{ik}.$$

If $\ell \neq k$ and then there is a unique i_0 such that $a_{i_0\ell} \neq 0$, but for this i_0 , $a_{i_0k} = 0$ by definition of H . Thus $c_\ell \cdot c_k = 0$ for all $\ell \neq k$, proving that the matrices in H are orthogonal. (2 marks) **Cat B Unseen but standard.**

- (ii) The matrices in H are those whose columns form a permutation of the standard basis. Indeed on the one hand, every matrix whose columns are a permutation of the standard basis is clearly in H . Conversely, let $A \in H$. Then, by definition, each of its columns is a vector of the standard basis, and since there is only one non-zero entry in each row, all the vectors of the standard basis appear as a column of H . It follows that there is $\sigma \in S_n$ such that

$$A = A_\sigma,$$

where $A_\sigma \in M_n(F)$ is the matrix whose i th column is $e_{\sigma(i)}$. Thus

$$H = \{A_\sigma \mid \sigma \in S_n\}.$$

(3 marks) **Cat C Unseen**

It is now easy to check that H is a subgroup. Clearly H is non-empty as it contains the identity matrix. Moreover let $\sigma \in S_n$. Then for $i \in \{1, \dots, n\}$

$$(A_\sigma)^{-1} e_{\sigma(i)} = e_i = A_{\sigma^{-1}} e_{\sigma(i)}.$$

Since $\{e_{\sigma(i)} \mid i = 1, \dots, n\}$ is a basis of $\mathbb{R}^n$ we deduce that $(A_\sigma)^{-1} = A_{\sigma^{-1}} \in H$. Finally, let $\sigma, \tau \in S_n$. Then for $i = 1, \dots, n$

$$(2) \quad A_\sigma A_\tau e_i = A_\sigma e_{\tau(i)} = e_{\sigma \circ \tau(i)} = A_{\sigma \circ \tau} e_i.$$

This implies that $A_\sigma A_\tau = A_{\sigma \circ \tau}$, because they coincide on the standard basis. Hence $A_\sigma A_\tau \in H$. This shows that H is a subgroup of G . (3 marks) **Cat D Unseen**

The map $\varphi : S_n \rightarrow H$ defined as $\sigma \mapsto A_\sigma$ is a homomorphism by (2). Clearly it is surjective because $H = \{A_\sigma \mid \sigma \in S_n\}$. Moreover $A_\sigma = I_n$ if and only if $\sigma = \text{id}$. Hence $\ker \varphi = \{\text{id}\}$ and so φ is an isomorphism. (1 mark) **Cat D Unseen**

- (6) (a) (i) Clearly $H \neq \emptyset$ as $(0, 0) \in H$. Let $x, y \in \mathbb{R}^2$. Then

$$(-x, -2x) \in H, \text{ and } (x, 2x) + (y, 2y) = (x + y, 2(x + y)) \in H.$$

Thus H is a subgroup by the subgroup test. (2 marks) **Cat A** Very Similar to Seen

- (ii) There is an isomorphism $\phi : H \rightarrow \mathbb{R}$ given by $(x, 2x) \rightarrow x$. Clearly this is a homomorphism as, for $x, y \in \mathbb{R}$,

$$\phi(x + y, 2(x + y)) = x + y = \phi(x, 2x) + \phi(y, 2y).$$

This has trivial kernel as $\phi(x, 2x) = 0$ if and only if $x = 0$. Moreover it is surjective because, for all $x \in \mathbb{R}$, $x = \phi(x, 2x)$. (2 marks) **Cat A** Very Similar to Seen.

- (iii) Let $a_1, b_1, a_2, b_2 \in \mathbb{R}$. We define

$$((a_1, b_1) + H) * ((a_2, b_2) + H) = (a_1 + a_2, b_1 + b_2) + H.$$

We check that $*$ is well defined and gives a group structure. Let $a'_1, b'_1, a'_2, b'_2 \in \mathbb{R}$ such that $(a_1 - a'_1, b_1 - b'_1) = (x, 2x) \in H$ and $(a_2 - a'_2, b_2 - b'_2) = (y, 2y) \in H$ for some $x, y \in \mathbb{R}$. Then

$$\begin{aligned} (a_1 + a_2, b_1 + b_2) + H &= \\ (a_1 - a'_1 + a'_1 + a_2 - a'_2 + a'_2, b_1 - b'_1 + b'_1 + b_2 - b'_2 + b'_2) + H &= \\ = (a'_1 + a'_2, b'_1 + b'_2) + (x + y, 2(x + y)) + H &= \\ = (a'_1 + a'_2, b'_1 + b'_2) + H. \end{aligned}$$

Clearly $*$ is a binary operation as $((a_1 + a_2, b_1 + b_2) + H)$ is a left coset of H for all $a_1, b_1, a_2, b_2 \in \mathbb{R}$. (2 marks) **Cat A** Similar to Seen

The operation is associative as it inherits associativity from the $+$ operation. The identity element is the coset $(0, 0) + H$. Indeed for $a, b \in \mathbb{R}$

$$((a, b) + H) * ((0, 0) + H) = ((a + 0, b + 0) + H) = ((a, b) + H) = ((0, 0) + H) * ((a, b) + H).$$

Finally, for all $a, b \in \mathbb{R}$,

$$\begin{aligned} ((a, b) + H) * ((-a, -b) + H) &= (a - a, b - b) + H = (0, 0) + H \\ &= ((-a, -b) + H) * ((a, b) + H). \end{aligned}$$

(2 marks) **Cat A** Similar to Seen

- (b) The decomposition is

$$(175436)(298).$$

To show this, we start by following the image of 1 through applying σ repeatedly: $1 \mapsto 7 \mapsto 5 \mapsto 4 \mapsto 3 \mapsto 6 \mapsto 1$. After this, we pick an element that is not moved by the cycle above, for instance 2, and follow its image through some repeated applications of σ until we land on 2 again: $2 \mapsto 9 \mapsto 8 \mapsto 2$.

(3 marks) **Cat B** Similar to Seen

- (c) (i) If the index of H in G is 2, then there is $g \in G$ such that

$$S_1 = \{H, gH\}.$$

Hence $gH = G \setminus H$ because the left cosets of H cover all of G .

There is a bijection between S_1 and S_2 given by $f : gH \rightarrow Hg^{-1}$: clearly f is well defined and injective because for all $g_1, g_2 \in G$, $g_1H = g_2H$ if and only if $g_2^{-1}g_1 \in H$, if and only if $Hg_2^{-1}g_1 = H$, if and only if $Hg_2^{-1} = Hg_1^{-1}$. Moreover f is also surjective because $Hg = f(g^{-1}H)$ for all $g \in G$.

(1 mark) **Cat B** Similar to Seen (Remark in the notes after def 2.9).

It follows that $S_2 = \{H, Hg'\}$ for some $g' \in G$ and hence $gH = G \setminus H = Hg'$ (because the right cosets cover all of G as they are equivalence classes of the equivalence $g_1 \sim g_2$ iff $g_2g_1^{-1} \in H$.)

(1 mark) **Cat B** Similar to what seen for left cosets.

- (ii) For this solution we view $\{1, -1\}$ as a subgroup of $\mathbb{Q}^\times$. By Theorem 4.18 in the notes the sgn map is a homomorphism $S_2 \rightarrow \{1, -1\}$; hence $H = \ker(\text{sgn})$.

(1 mark) **Cat C** Seen

Moreover if $\sigma \in S_5$ and $\text{sgn}(\sigma) = -1$, then $\text{sgn}(\sigma H) = -1$, because $\text{sgn}(\sigma\tau) = \text{sgn}(\sigma)\text{sgn}(\tau) = -1 \cdot 1 = -1$. Thus H has only two left cosets in S_5 and $|S_5 : H| = 2$.

We conclude that the set of left cosets is the same as the set of right cosets by the previous point.

(2 marks) **Cat C** Unseen

- (d) Let n be the order of G . For all $g \in G$ we define

$$\lambda_g : G \rightarrow G \text{ by } h \mapsto gh.$$

We claim that λ_g is a bijection for all $g \in G$. Indeed, for all $k \in G$, $k = g(g^{-1}k) = \lambda_g(g^{-1}k)$. Thus λ_g is surjective. It is also injective, because if $\lambda_g(h) = \lambda_g(k)$ for some $h, k \in G$, we deduce that

$$gh = gk; \text{ hence } h = k \text{ by multiplying the last identity by } g^{-1} \text{ on the left.}$$

The rule $g \mapsto \lambda_g$ defines a map

$$\Lambda : G \rightarrow \text{Sym}(G).$$

We show that Λ is an injective homomorphism. This will conclude the proof as then the image of Λ will be a subgroup of $\text{Sym}(G)$ isomorphic to G and $\text{Sym}(G)$ is isomorphic to S_n .

We prove that Λ is a homomorphism. Let $g, g' \in G$. Then for all $h \in G$,

$$\lambda_{gg'}(h) = (gg')h = g(g'h) = g\lambda_{g'}(h) = \lambda_g(\lambda_{g'}(h)) = \lambda_g \circ \lambda_{g'}(h).$$

This shows that Λ is a homomorphism because the product in $\text{Sym}(G)$ is the composition of functions. It remains to show that the map is injective. To this end, let $g, g' \in G$ be such that $gh = g'h$ for all $h \in G$. Then, multiplying on the right by h^{-1} we get $g = g'$; thus Λ is injective.

(4 marks) **Cat D** Unseen.

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
		"Part (a) had a very elegant solution, if you preformed the row operations while keeping entry of the augmented matrix as integers, then the result was valid for all 3 parts, and would lead to a very quick solution. It was much more common to do 3 separate row reductions, which is a perfectly valid method.
Linear Algebra and Groups_MATH40003	1	You must check that the subspace test can be applied every time you use it. This was a very common mistake."
Linear Algebra and Groups_MATH40003	2	
		(c) was extremely challenging, well done to anyone who got it right. There are lots of possible solutions to it, and the method presented in the solutions was the rarest one to be actually used in the exam.
		The most common idea was to consider $\ker(T)$, $\ker(T^2)$, ... showing that this is an increasing chain of subspaces of V . The point at which this chain becomes V is closely linked to the degree of T . This method needs a lot of lemmas to work, however, so was hard to get full marks with.
Linear Algebra and Groups_MATH40003	3	Another method used the Cayley-Hamilton Theorem. It's possible (but not easy) to show that the characteristic polynomial of T must be x^i where $i < \dim(V)$. You then needed to show that $\deg(T) < i$.
Linear Algebra and Groups_MATH40003	4	
Linear Algebra and Groups_MATH40003	5	

5a was generally solved well (there is no need to show that a set of orthogonal vectors is linearly independent because this is always the case, so I did not penalise those that did not show that)

5b not everyone produced a full proof of the shape of the rotation matrix but I have been lenient marking this one

Most students used a different argument for 5c to the one in the solutions, I have therefore used a different marking scheme:

3 marks for subgroup test (1 each part)

6 4 marks for isomorphism (1 for map 1 for homomorphism 1 for injective 1 for surjective)