

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Linear Algebra and Groups

Date: 7th May 2020

Time: 09.00am – 12.00 noon (BST)

Time Allowed: 3 Hours

Upload Time Allowed: 30 Minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Let A be the following 3×3 matrix over $\mathbb{R}$:

$$A := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

- (a) Compute A^{-1} using elementary row operations. (2 marks)
- (b) Define what it means for an $n \times n$ matrix to be orthogonal. (1 mark)
- (c) Show A is orthogonal. (1 mark)
- (d) Let R_θ be the anti-clockwise rotation of $\mathbb{R}^2$ about the origin through θ radians. Show that R_θ can be represented by the matrix:

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

You may assume R_θ is linear.

(4 marks)

- (e) Recall that a reflection M_θ in the line $L_\theta = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1 \tan \theta\}$ has matrix representation

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Let M be the reflection of the plane $\mathbb{R}^2$ in the x_1 -axis, and M' the reflection in the line $x_1 = x_2$. Write down the matrices representing M , M' and show that MM' is a rotation.

(3 marks)

- (f) Show that all real orthogonal 2×2 matrices are either rotations or reflections. (7 marks)

- (g) Let $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an anticlockwise rotation by θ around the

axis $\{l \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : l \in \mathbb{R}\}$. Suppose $B := \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right)$ is an orthonormal basis for $\mathbb{R}^3$.

Find the possibilities for $[T]_B$.

(2 marks)

(Total: 20 marks)

2. Consider the set of equations, where $a, b \in \mathbb{R}$:

$$\left. \begin{aligned} x + 2y - z + 3w &= 0 \\ 2x - 3z + 4w &= 1 \\ x + 4y + z + aw &= b \\ -2y - w &= 0 \end{aligned} \right\}.$$

- (a) Find the augmented matrix $(A|c)$ for this set of linear equations. (1 mark)
- (b) For which values of $a, b \in \mathbb{R}$ does this set of equations have a solution? Find all solutions when possible. (6 marks)
- (c) (i) Define the row space and row rank of a matrix $B \in M_{n \times m}(\mathbb{R})$. (1 mark)
- (ii) Suppose $(A|c)$ is the augmented matrix obtained from the system of equations above. In the case where $a = 4$ and $b = 2$ find the row space and rank of A and $(A|c)$. (4 marks)
- (d) Given a system of n real linear equations in n variables (not necessarily the one above) with augmented matrix $(A|c)$. Show that if the row rank of A equals the row rank of $(A|c)$ then the set of solutions S is a coset of a subspace $K \subseteq \mathbb{R}^n$ where the dimension of K is $n - \text{rank}(A)$.
In part (d) you may assume standard results about matrices in row reduced echelon form, and linear transformations (including the rank-nullity theorem). (8 marks)

(Total: 20 marks)

3. Let V and W be vector spaces over $\mathbb{R}$.

(a) Let U be a subset of V . Give conditions for U to be a subspace of V . (3 marks)

(b) Let $V = \mathbb{R}[x]$ the set of polynomials with real coefficients and variable x . For a fixed $n \in \mathbb{N}$ let

$$U_n = \{f(x) \in \mathbb{R}[x] : f(x) = \sum_{i=0}^n a_i x^i, a_{i+1} = 2a_i \text{ for } i \in \{0, 1, 2, \dots, n-1\}\}.$$

Show that U_n is a subspace of V . Find a basis for U_n . (4 marks)

(c) Let $T : V \rightarrow W$ be a map from V to W .

- * Give conditions for T to be a linear transformation.
- * Define the kernel of T , $\ker(T)$.
- * Define the image of T , $\text{Im}(T)$.
- * Show that $\ker(T)$ is a subspace of V .

(8 marks)

(d) Let V and U_n be as in part (b). Let $T_n : U_n \rightarrow U_{n+1}$ be defined as follows:

$$T_n\left(\sum_{i=0}^n a_i x^i\right) = \sum_{i=0}^{n+1} 2^i a_0 x^i.$$

- * Show that T_n is a linear transformation.
- * Find $\text{Im}(T)$.
- * Find $\ker(T)$.
- * Is U_n a subspace of U_{n+1} ?

(5 marks)

(Total: 20 marks)

4. (a) Suppose $n \in \mathbb{N}$ and $A \in M_n(\mathbb{R})$ is such that $A^2 - 3A + 2I_n = 0_n$ (where 0_n denotes the $n \times n$ zero matrix).
- (i) Prove that A is invertible and give an expression for A^{-1} . (3 marks)
- For $\lambda = 1, 2$, let $E_\lambda = \{v \in \mathbb{R}^n : Av = \lambda v\}$.
- (ii) Prove that $E_1 \cap E_2 = \{0\}$ and $E_1 + E_2 = \mathbb{R}^n$. (3 marks)
- (iii) Prove that A is diagonalisable. (2 marks)
- (iv) Suppose $v \in \mathbb{R}^n$ and $\{\|A^k v\| : k \in \mathbb{N}\}$ is bounded (where $\|\cdot\|$ is the usual norm on $\mathbb{R}^n$). Prove that $v \in E_1$. (2 marks)
- (v) In the case $n = 2$ write down three matrices in $M_2(\mathbb{R})$ such that for any $A \in M_2(\mathbb{R})$ which satisfies $A^2 - 3A + 2I_2 = 0$, there is an invertible $P \in M_2(\mathbb{R})$ such that $P^{-1}AP$ is equal to one of these three matrices. Explain your answer briefly. (2 marks)
- (b) Suppose $B \in M_n(\mathbb{R})$ is invertible and let $C = B^T B$.
- (i) Prove that for every non-zero vector $v \in \mathbb{R}^n$ we have $v^T C v > 0$. (2 marks)
- (ii) Show that there exist orthogonal matrices $P, Q \in M_n(\mathbb{R})$ and a diagonal matrix $D \in M_n(\mathbb{R})$ with $B = QDP$.
[You may use results from the lectures if they are quoted precisely.] (6 marks)

(Total: 20 marks)

5. (a) State the definition of the *determinant* $\det(A)$ of an $n \times n$ matrix A over a field F . Suppose $\alpha \in F$ and $A \in M_n(F)$. Let $i \leq n$ and let $B \in M_n(F)$ be the matrix which results from multiplying row i of A by α (and leaving all the other rows the same). Using your definition, prove that $\det(B) = \alpha \det(A)$.

(6 marks)

- (b) Compute $\det(X^2 Y^3 X^T Z)$ where $X, Y, Z \in M_3(\mathbb{R})$ are given by:

$$X = \begin{pmatrix} 1 & 1 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad Z = \begin{pmatrix} 0 & 1 & 4 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

(3 marks)

- (c) Suppose H is a subgroup of the group $(G, \cdot)$.

- (i) Define what is meant by a *left coset* of H in G .

(2 marks)

Suppose that X, Y are left cosets of H in G .

- (ii) Prove that either $X \cap Y = \emptyset$ or $X = Y$.

(3 marks)

- (iii) Prove that there is a bijection $X \rightarrow Y$.

(2 marks)

- (iv) Suppose that K is also a subgroup of G and Z is a left coset of K in G . Prove that either $X \cap Z = \emptyset$ or $X \cap Z$ is a left coset of $H \cap K$ in G . Give examples to show that both of these possibilities can occur.

(4 marks)

(Total: 20 marks)

6. (a) Suppose $(G, \cdot)$ is a group and $g \in G$.

(i) Define the *order* of g .

Suppose G is a cyclic group of order 12.

(ii) Give an example of such a group G .

(iii) Determine the possible orders of elements of G and say how many elements of each order there are in G .

(iv) Find subgroups A, B of G such that $A, B \neq G$, $A \cap B = \{e\}$ and $G = \{ab : a \in A, b \in B\}$. Justify your answer.

(9 marks)

(b) Suppose G, H are groups and $\phi : G \rightarrow H$ is a homomorphism.

(i) Define the term *homomorphism* and say what is meant by the *kernel*, $\ker(\phi)$, of ϕ .

(2 marks)

(ii) Prove that $N = \ker(\phi)$ is a subgroup of G and for all $g_1, g_2 \in G$ we have:

$$\phi(g_1) = \phi(g_2) \Leftrightarrow g_1 N = g_2 N.$$

(4 marks)

(c) For each of the following pairs of groups G, H , find a *non-trivial* homomorphism $\phi : G \rightarrow H$ (non-trivial here means that there is $g \in G$ with $\phi(g) \neq e_H$ and, if $G = H$, then ϕ is not the identity map.).

(i) $G = H = GL_2(\mathbb{R})$;

(ii) $G = (\mathbb{R}, +)$ and $H = (\mathbb{C}^\times, \cdot)$;

(iii) $G = GL_2(\mathbb{R})$ and $H = (\mathbb{R}^\times, \cdot)$;

(iv) $G = (\mathbb{R}, +)$ and $H = GL_2(\mathbb{R})$.

(5 marks)

(Total: 20 marks)

1. Let A be the following 3×3 matrix over $\mathbb{R}$:

$$A := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) Compute A^{-1} using elementary row operations.

(2 marks)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & | & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & | & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 1 & 0 & 1 & | & 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & | & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 2 & | & -\sqrt{2} & 0 & \sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & | & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & | & \sqrt{2} - \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

As $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ we get

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

2 marks (A)

- (b) Define what it means for an $n \times n$ matrix to be orthogonal. (1 mark)

An invertible matrix A is orthogonal if $A^{-1}A^T = I_n$ (and therefore A is invertible, with $A^{-1} = A^T$). 1 mark (A)

- (c) Show A is orthogonal. (1 mark)

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = A^T$$

1 mark (A)

- (d) Let R_θ be the anti-clockwise rotation of $\mathbb{R}^2$ about the origin through θ radians. Show that R_θ can be represented by the matrix:

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

You may assume R_θ is linear.

(4 marks)

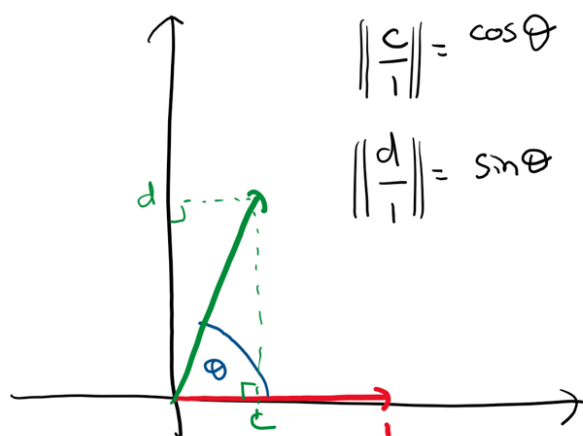
Let $x \in \mathbb{R}^2$ then $x = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ now as R_θ is linear we have

$R_\theta x = R_\theta \begin{pmatrix} a \\ b \end{pmatrix} = a R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, thus by distributivity of matrix

multiplication it is sufficient to find a matrix that correctly deals with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

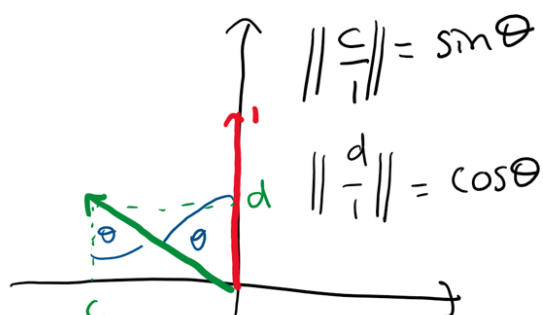
and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

1 mark (A)



So

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$



So

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}.$$

3 mark (B)

- (e) Recall that a reflection M_θ in the line $L_\theta = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1 \tan \theta\}$ has matrix representation

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Let M be the reflection of the plane $\mathbb{R}^2$ in the x_1 -axis, and M' the reflection in the line $x_1 = x_2$. Write down the matrices representing M , M' and show that MM' is a rotation.

(3 marks)

To work out what the matrix does one could either calculate $\cos 2\theta$ and $\sin 2\theta$ in both cases and plug the values in, or consider what happens to the basis vectors e_1 and e_2 ,

* $Me_1 = e_1, Me_2 = -e_2$

* $M'e_1 = e_2, M'e_2 = e_1$

2 marks (A)

Thus

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Multiplying gives $MM' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = R_\theta$ where $\theta = \frac{3\pi}{2}$. 1 mark (A)

- (f) Show that all real orthogonal 2×2 matrices are either rotations or reflections. (7 marks)

There are a few ways of doing this, here is one:

As $AA^T = I_n$ we get $1 = \det(I_n) = \det(AA^T) = \det(A)\det(A^T) = \det(A)^2$ therefore $\det A = \pm 1$. 1 mark (C)

Case 1: $\det A = 1$ So we have

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Now we get $a = d$ and $c = -b$ thus we get $a^2 + b^2 = 1$, so as $a^2, b^2 \geq 0$ we get $a^2, b^2 \leq 1$ thus $-1 \leq a, b \leq 1$. So there is a $\theta \in [0, 2\pi]$ such that $\cos \theta = a$, so we get:

$$b^2 = 1 - \cos^2 \theta = \sin^2 \theta \text{ so } b = \pm \sin \theta$$

Suppose $b = \sin \theta$ then we get:

$$A = \begin{pmatrix} \cos \theta & \sin \theta \\ \cos \theta & -\sin \theta \end{pmatrix} = R_\theta$$

Suppose $b = -\sin \theta$ then $b = \sin(2\pi - \theta)$ and $a = \cos \theta = \cos(2\pi - \theta)$ then we get:

$$A = \begin{pmatrix} \cos(2\pi - \theta) & \sin(2\pi - \theta) \\ \cos(2\pi - \theta) & -\sin(2\pi - \theta) \end{pmatrix} = R_{(\pi - \theta)}$$

In both cases we get a rotation. 3 marks (C)

Case 2: $\det A = -1$ So we have

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

Now we get $a = -d$ and $c = b$ thus we get $-(a^2 + b^2) = -1$, so as $a^2, b^2 \geq 0$ we get $a^2, b^2 \leq 1$ thus $-1 \leq a, b \leq 1$. So there is a $\theta \in [0, 2\pi]$ such that $\cos 2\theta = a$, so we get:

$$b^2 = 1 - \cos^2(2\theta) = \sin^2(2\theta) \text{ so } b = \pm \sin(2\theta)$$

Suppose $b = \sin(2\theta)$ then we get:

$$A = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \cos(2\theta) & -\sin(2\theta) \end{pmatrix} = M_\theta$$

Suppose $b = -\sin(2\theta)$ then $b = \sin(2\pi - 2\theta)$ and $a = \cos(2\theta) = \cos(2\pi - 2\theta)$ then we get:

$$A = \begin{pmatrix} \cos(2\pi - 2\theta) & \sin(2\pi - 2\theta) \\ \cos(2\pi - 2\theta) & -\sin(2\pi - 2\theta) \end{pmatrix} = M_{(\pi-\theta)}$$

In both cases we get a reflection.

3 mark (C)

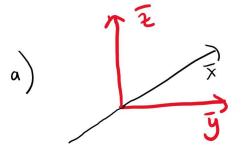
- (g) Let $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an anticlockwise rotation by θ around the axis $\{l \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : l \in \mathbb{R}\}$. Suppose $B := \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right)$ is an orthonormal basis for $\mathbb{R}^3$.

Find the possibilities for $[T]_B$.

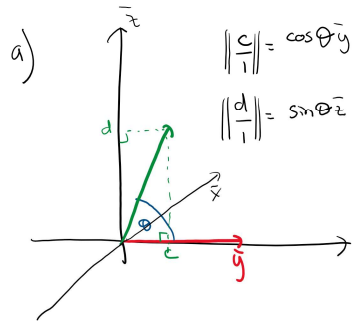
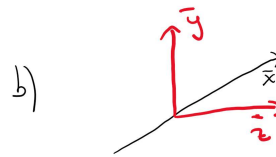
(2 marks)

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1 \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 0 \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + 0 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

There are two possibilities for $\bar{y}$ and $\bar{z}$



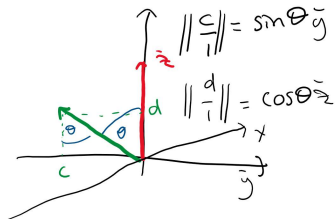
or



$$\left\| \frac{c}{1} \right\| = \cos \theta \bar{y}$$

$$\left\| \frac{d}{1} \right\| = \sin \theta \bar{z}$$

so $\bar{y} \mapsto 0\bar{x} + \cos \theta \bar{y} + \sin \theta \bar{z}$



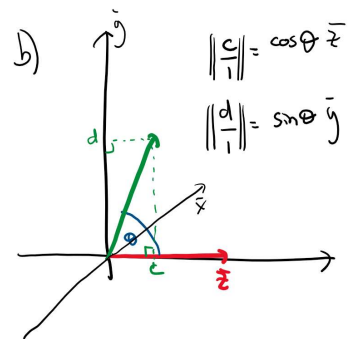
$$\left\| \frac{c}{1} \right\| = \sin \theta \bar{y}$$

$$\left\| \frac{d}{1} \right\| = \cos \theta \bar{z}$$

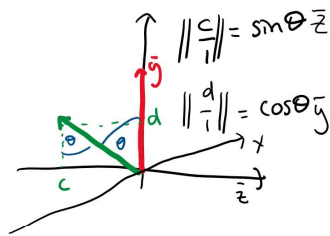
so $\bar{z} \mapsto 0\bar{x} - \sin \theta \bar{y} + \cos \theta \bar{z}$

so
$$[T]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

1 mark (C)



So $\bar{z} \mapsto 0\bar{x} + \sin \theta \bar{y} + \cos \theta \bar{z}$



So $\bar{y} \mapsto 0\bar{x} + \cos \theta \bar{y} - \sin \theta \bar{z}$

So $[T]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin \theta & \cos \theta \\ 0 & \cos \theta & -\sin \theta \end{pmatrix}$

1 mark (C)

Comments: Parts (a)-(e) either bookwork or similar exercises on problem

sheets (f) and (g) are unseen though should not be too challenging.

(Total: 20 marks)

2. Consider the set of equations, where $a, b \in \mathbb{R}$:

$$\left. \begin{aligned} x + 2y - z + 3w &= 0 \\ 2x - 3z + 4w &= 1 \\ x + 4y + z + aw &= b \\ -2y - w &= 0 \end{aligned} \right\}.$$

(a) Find the augmented matrix $(A|c)$ for this set of linear equations.

(1 mark)

The Augmented matrix is the matrix:

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 2 & 0 & -3 & 4 & 1 \\ 1 & 4 & 1 & a & b \\ 0 & -2 & 0 & -1 & 0 \end{array} \right)$$

1 mark (A)

(b) For which values of $a, b \in \mathbb{R}$ does this set of equations have a solution? Find all solutions when possible.

(6 marks)

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 2 & 0 & -3 & 4 & 1 \\ 1 & 4 & 1 & a & b \\ 0 & -2 & 0 & -1 & 0 \end{array} \right) \mapsto \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & -4 & -1 & -2 & 1 \\ 0 & 2 & 2 & a-3 & b \\ 0 & -2 & 0 & -1 & 0 \end{array} \right)$$

$$\mapsto \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & a-4 & b \\ 0 & -2 & 0 & -1 & 0 \end{array} \right)$$

$$\begin{aligned}
& \mapsto \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & a-4 & b+2 \end{array} \right) \\
& \mapsto \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & a-4 & b+2 \end{array} \right) \quad \dagger \\
& \text{when } a \neq 4 \mapsto \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & \frac{b+2}{a+4} \end{array} \right) \\
& \mapsto \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 - \frac{2b+4}{a+4} \\ 0 & -2 & 0 & 0 & \frac{b+2}{a+4} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & \frac{b+2}{a+4} \end{array} \right)
\end{aligned}$$

3 marks (A)

So we get the following:

* When $a \neq 4$

$$x = -1 - \frac{2b+4}{a+4}, \quad y = -\frac{b+2}{2a+8}, \quad z = -1, \quad w = \frac{b+2}{a+4}$$

1 mark (A)

* When $a = 4$ we have:

- If $b \neq 2$ we have $b-2 \neq 0$ so from the last line of $\dagger$ we get $0 = b_2 \neq 0$ so we get a contradiction therefore no solutions. 1 mark (A)
- If $b = 2$ we have the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Thus

$$x + 2w = -1 \quad \text{so} \quad x = -2w - 1$$

$$-2y - w = 0 \quad \text{so} \quad y = \frac{w}{2} - z = 1$$

So we get the set $\left\{ \begin{pmatrix} -2t - 1 \\ \frac{t}{2} \\ -1 \\ t \end{pmatrix} \in \mathbb{R}^4 : t \in \mathbb{R} \right\}$ 1 mark (A)

- (c) (i) Define the row space and row rank of a matrix $B \in M_{n \times m}(\mathbb{R})$. (1 mark)

The Row Space of A ($RSp(B)$) is the span of the rows of A . The Row Rank of A is $\dim(RSp(B))$. 1 mark (A)

- (ii) Suppose $(A|c)$ is the augmented matrix obtained from the system of equations above. In the case where $a = 4$ and $b = 2$ find the row space and rank of A and $(A|c)$.

(4 marks)

First note that row operations preserve the row space.

- (iii) **The row space of A is equal to the row space of:**

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus the row space of $(A|b)$ is

$$Span((1 \ 0 \ 0 \ 2), (0 \ -2 \ 0 \ -1), (0 \ 0 \ -1 \ 0), (0 \ 0 \ 0 \ 0))$$

Now obviously $\{(1 \ 0 \ 0 \ 2), (0 \ -2 \ 0 \ -1), (0 \ 0 \ -1 \ 0)\}$ spans the row space of A , and is linearly independent, so the rank of $(A|c)$ is 3. 2 marks (C)

- * The row space of $(A|c)$ is equal to the row space of:**

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Thus the row space of $(A|b)$ is

$$Span((1 \ 0 \ 0 \ 2 \ -1), (0 \ -2 \ 0 \ -1 \ 0), (0 \ 0 \ -1 \ 0 \ 1), (0 \ 0 \ 0 \ 0 \ 0))$$

Now obviously $\{ (1 \ 0 \ 0 \ 2 \ -1), (0 \ -2 \ 0 \ -1 \ 0), (0 \ 0 \ -1 \ 0 \ 1) \}$ spans the row space of $(A|c)$, and is linearly independent, so the rank of $(A|c)$ is 3. 2 marks (C)

- (d) Given a system of n real linear equations in n variables (not necessarily the one above) with augmented matrix $(A|c)$. Show that if the row rank of A equals the row rank of $(A|c)$ then the set of solutions S is a coset of a subspace $K \subseteq \mathbb{R}^n$ where the dimension of K is $n - \text{rank}(A)$.

In part (d) you may assume standard results about matrices in row reduced echelon form, and linear transformations (including the rank-nullity theorem). (8 marks)

Consider the row reduced matrices A_{ech} and $(A|c)_{ech}$ (which we will denote $(A_{ech}|c_{ech})$), suppose the last k rows of A_{ech} are zero, thus (as the row rank of A equals the row rank of $(A|c)$) the last k entries of c_{ech} are 0 (where $0 \leq k \leq n$).

Now by construction the set of solutions to the system of linear equations equals the set of solutions to $A_{ech}x = c_{ech}$ where x is a vector of length n . Thus it is sufficient to show the above where $(A|c)$ is in row reduced echelon form. 2 marks (D)

We can consider $A_{ech} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as a linear transformation. The Image of A_{ech} is spanned by the set of vectors $\{A_{ech}e_1, \dots, A_{ech}e_n\}$ where $e_1, \dots, e_n$ are the standard basis for $\mathbb{R}^n$. Now $A_{ech}e_i$ is just the i^{th} column of A_{ech} , so the columns of A_{ech} span $Im(A_{ech})$. Now as A_{ech} is in row reduced echelon form it must contain columns of the form $e_1, \dots, e_{n-k}$. Now as the last k entries of c_{ech} are 0 we have that $c_{ech} \in Span(A_{ech}e_1, \dots, A_{ech}e_n)$ and thus $c_{ech} \in Im(A_{ech})$. Thus we have a $c_1 \in \mathbb{R}^n$ such that $A_{ech}c_1 = c_{ech}$. 3 marks (D)

Claim: $S = \ker(A_{ech}) + c_1$

Proof of claim: Let $x \in S$ consider $x - c_1$, as A_{ech} is linear $A_{ech}(x - c_1) = A_{ech}x - A_{ech}c_1 = c - c = 0$, thus $x - c_1 \in \ker(A_{ech})$ so $x \in \ker(A_{ech}) + c_1$.

Now let $x \in \ker(A_{ech}) + c_1$ then $x = z + c_1$ where $z \in \ker(A_{ech})$ so $A_{ech}x = A_{ech}(z + c_1) = A_{ech}z + A_{ech}c_1 = 0 + c = c$ thus $c \in S$.

Now by the rank nullity theorem $\ker(A_{ech})$ is a subspace of $\mathbb{R}^n$ of dimension $n - \dim(Im(A_{ech})) = n - \text{rank}(A_{ech})$. Thus by putting $K = \ker(A_{ech})$ we have the desired result. 3 marks (D)

Comments: part (d) unseen (a)-(c) bookwork or similar seen on problem sheets.

(Total: 20 marks)

3. Let V and W be vector spaces over $\mathbb{R}$.

(a) Let U be a subset of V . Give conditions for U to be a subspace of V . (3 marks)

A subset W of a vector space V is a subspace of V if

S1 W is not empty (i.e. $e \in W$) 1 mark (A)

S2 for $v, w \in W$, then $v \oplus w \in W$ closed under vector addition 1 mark (A)

S3 $v \in W$ and $r \in \mathbb{R}$, then $r \odot v \in W$ closed under scalar multiplication. 1 mark (A)

(b) Let $V = \mathbb{R}[x]$ the set of polynomials with real coefficients and variable x . For a fixed $n \in \mathbb{N}$ let

$$U_n = \{f(x) \in \mathbb{R}[x] : f(x) = \sum_{i=0}^n a_i x^i, a_{i+1} = 2a_i \text{ for } i \in \{0, 1, 2, \dots, n-1\}\}.$$

Show that U_n is a subspace of V . Find a basis for U_n . (4 marks)

It is clear that $U \subset V$ so we can apply the subspace test: 1 mark (B)

S1 $0 + 0x + \dots 0x^n = \sum_{i=0}^n 0x^i$ and $0 = 2 \cdot 0$. So $0 + 0x \in U_n$, thus U_n is non-empty. 1 mark (B)

S2 Let $f(x), g(x) \in U_n$ then $g(x) = \sum_{i=0}^n a_i x^i$ with $a_{i+1} = 2a_i$ and $f(x) = \sum_{i=0}^n b_i x^i$ with $b_{i+1} = 2b_i$ for $i \in \{0, 1, 2, \dots, n-1\}$ then $f(x) + g(x) = \sum_{i=0}^n (a_i + b_i)x^i$ and $a_{i+1} + b_{i+1} = 2a_i + 2b_i = 2(a_i + b_i)$ for $i \in \{0, 1, 2, \dots, n-1\}$, so the condition is satisfied. 1 mark (B)

S3 Let $g(x) \in U_n$, $\lambda \in \mathbb{R}$ then $g(x) = \sum_{i=0}^n a_i x^i$ with $a_{i+1} = 2a_i$, so $\lambda g(x) = \lambda \sum_{i=0}^n a_i x^i = \sum_{i=0}^n (\lambda a_i)x^i$. Now we have that $\lambda a_{i+1} = 2\lambda a_i$ so the condition is satisfied. 1 mark (B)

(c) Let $T : V \rightarrow W$ be a map from V to W .

* Give conditions for T to be a linear transformation. **T is a linear transformation if it:**

• **preserves addition, i.e. if for all $v_1, v_2 \in V$ we have $T(v_1 + v_2) = T(v_1) + T(v_2)$. (i.e. if $T(v_1) = w_1$, $T(v_2) = w_2$ for $w_1, w_2 \in W$ we have $T(v_1 + v_2) = w_1 + w_2$.** 1 mark (A)

• **preserves scalar multiplication, i.e. if for all $v \in V$, $\lambda \in F$, $T(\lambda v) = \lambda T(v)$** 1 mark (A)

* Define the kernel of T , $\ker(T)$. **The Kernel of T is the set $\ker T = \{v \in V : T(v) = 0_W\} \subseteq V$.** 1 mark (A)

* Define the image of T , $\text{Im}(T)$. **The Image of T is the set $\text{Im} T = \{T(v) \in W : v \in V\} \subseteq W$.** 1 mark (A)

* Show that $\ker(T)$ is a subspace of V . **It is clear that $\ker T$ is a subset of V so we can apply the subspace test:** 1 mark (B)

S1 For $0_V \in V$ consider $T(0_V) = T(0_V - 0_V) = T(0_V) - T(0_V) = 0_W$ so $0_V \in \ker T$. 1 mark (B)

S2 Suppose $v_1, v_2 \in \ker T$ then $T(v_1 + v_2) = T(v_1) + T(v_2) = 0_W + 0_W = 0_W$, so $v_1 + v_2 \in \ker T$. 1 mark (B)

S3 Let $v \in \ker T$ and $\lambda \in \mathbb{R}$ then $T(\lambda v) = \lambda T(v) = \lambda 0_W = 0_W$ so $\lambda v \in \ker T$.

1 mark (B)

Therefore $\ker T$ is a subspace of V

(8 marks)

(d) Let V and U_n be as in part (b). Let $T_n : U_n \rightarrow U_{n+1}$ be defined as follows:

$$T_n\left(\sum_{i=0}^n a_i x^i\right) = \sum_{i=0}^{n+1} 2^i a_0 x^i.$$

* Show that T_n is a linear transformation. **Let $f(x), g(x) \in U_n$ and $\lambda \in \mathbb{R}$ then $g(x) = \sum_{i=0}^n a_i x^i$ with $a_{i+1} = 2a_i$ and $f(x) = \sum_{i=0}^n b_i x^i$ with $b_{i+1} = 2b_i$ for $i \in \{0, 1, 2, \dots, n-1\}$**

· **$T(g(x) + f(x)) = \sum_{i=0}^{n+1} 2^i (a_0 + b_0) x^i = \sum_{i=0}^{n+1} 2^i a_0 x^i + \sum_{i=0}^{n+1} 2^i b_0 x^i = T(g(x)) + T(f(x))$. So T preserves addition.**

1 mark (D)

· **$T(\lambda g(x)) = \sum_{i=0}^{n+1} 2^i (\lambda a_0) x^i = \lambda \sum_{i=0}^{n+1} 2^i a_0 x^i = \lambda T(g(x))$. So T preserves scalar multiplication.**

1 mark (D)

* Find $\text{Im}(T)$. **First notice that if $g(x) = \sum_{i=0}^{n+1} a_i x^i \in U_{n+1}$ then for each $i = 0, \dots, n+1$ we have $a_{i+1} = 2a_i = 2^2 a_{i-1} = \dots = 2^i a_0$, that is to say that $g(x)$ is completely determined by a_0 . Now given a $g(x) = \sum_{i=0}^{n+1} a_i x^i \in U_{n+1}$ we have $f(x) = \sum_{i=0}^n 2^i a_0 x^i \in U_n$ and $T(f(x)) = \sum_{i=0}^{n+1} 2^i a_0 x^i = g(x)$. Thus $\text{Im}(T) = U_{n+1}$.**

1 mark (D)

* Find $\ker(T)$. **Let $g(x) = \sum_{i=0}^n a_i x^i \in U_n$ so $a_{i+1} = 2a_i$**

$$\begin{aligned} \ker T &= \{g(x) \in U_n : T(g(x)) = 0_W\} \\ &= \{g(x) \in U_n : \sum_{i=0}^{n+1} 2^i a_0 x^i = 0, \} \\ &= \{g(x) \in U_n : 2^i a_0 = 0 \text{ for } i = 1, \dots, n+1\} \\ &= \{g(x) \in U_n : a_0 = 0\} \end{aligned}$$

Now $a_i = 2a = 2^i a_0$ we get that if $g(x) = \sum_{i=0}^n a_i x^i \in U_n$ with $a_0 = 0$ then $a_i = 0$ for $i = 1, \dots, n$. Thus $\ker T = \{\sum_{i=0}^n 0x^i\}$

1 mark (D)

* Is U_n a subspace of U_{n+1} ? **No - because it is not a subset of U_{n+1}**

1 mark (A)

(5 marks)

Comments: Examples are unseen and a bit different from examples on problem sheets. Definitions all bookwork.

(Total: 20 marks)

4. (a) Suppose $n \in \mathbb{N}$ and $A \in M_n(\mathbb{R})$ is such that $A^2 - 3A + 2I_n = 0_n$ (where 0_n denotes the $n \times n$ zero matrix).
- (i) Prove that A is invertible and give an expression for A^{-1} . (3 marks)
- For $\lambda = 1, 2$, let $E_\lambda = \{v \in \mathbb{R}^n : Av = \lambda v\}$.
- (ii) Prove that $E_1 \cap E_2 = \{0\}$ and $E_1 + E_2 = \mathbb{R}^n$. (3 marks)
- (iii) Prove that A is diagonalisable. (2 marks)
- (iv) Suppose $v \in \mathbb{R}^n$ and $\{\|A^k v\| : k \in \mathbb{N}\}$ is bounded (where $\|\cdot\|$ is the usual norm on $\mathbb{R}^n$). Prove that $v \in E_1$. (2 marks)
- (v) In the case $n = 2$ write down three matrices in $M_2(\mathbb{R})$ such that for any $A \in M_2(\mathbb{R})$ which satisfies $A^2 - 3A + 2I_2 = 0$, there is an invertible $P \in M_2(\mathbb{R})$ such that $P^{-1}AP$ is equal to one of these three matrices. Explain your answer briefly. (2 marks)
- (b) Suppose $B \in M_n(\mathbb{R})$ is invertible and let $C = B^T B$.
- (i) Prove that for every non-zero vector $v \in \mathbb{R}^n$ we have $v^T C v > 0$. (2 marks)
- (ii) Show that there exist orthogonal matrices $P, Q \in M_n(\mathbb{R})$ and a diagonal matrix $D \in M_n(\mathbb{R})$ with $B = QDP$.
[You may use results from the lectures if they are quoted precisely.] (6 marks)
- (a) (i) Let $B = \frac{1}{2}(3I_n - A)$. By the given equation $AB = BA = I_n$, so A is invertible, with inverse B . 3 marks (C)
- (ii) If $v \in E_1 \cap E_2$ then $Av = v$ and $Av = 2v$, thus $v = 0$.
Note that the given equation is $(A - 2I_n)(A - I_n) = 0_n$. Moreover, for any $v \in \mathbb{R}^n$ we have $v = (A - I_n)v - (A - 2I_n)v$. So if $v_1 = (A - I_n)v$ and $v_2 = (A - 2I_n)v$, then $(A - 2I_n)v_1 = 0$ (by the equation) and similarly $(A - I_n)v_2 = 0$. So $v_1 \in E_2$ and $v_2 \in E_1$. Thus $E_1 + E_2 = \mathbb{R}^n$. 3 marks (B)
- (iii) Let B be the union a basis of E_1 and a basis of E_2 . By (ii), B is a basis of $\mathbb{R}^n$. So B is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A : thus A is diagonalisable. 2 marks (B)
- (iv) Write $v = v_1 + v_2$ where $v_i \in E_i$. Then $A^k v = v_1 + 2^k v_2$ and $\|A^k v\| \geq 2^k \|v_2\| - \|v_1\|$ (by the triangle inequality). By the boundedness, we therefore have $\|v_2\| = 0$. Thus $v_2 = 0$ and $v = v_1 \in E_1$. 2 marks (C)
- (v) The three cases correspond to the possible dimensions of E_1 and E_2 . If E_2 is the zero subspace, then $E_1 = \mathbb{R}^2$ and $A = I_2$. Similarly if E_1 then $A = 2I_2$. In the remaining case, 1, 2 are both eigenvalues of A and so there is an invertible P such that $P^{-1}AP$ is the diagonal matrix $\text{diag}(1, 2)$. 2 marks (A)
- (b) (i) We have $v^T C v = v^T B^T B v = (Bv)^T (Bv) = \|Bv\|^2$. This is ≥ 0 and is equal to 0 iff $Bv = 0$. As B is invertible, this happens iff $v = 0$. 2 marks (A)
- (ii) The matrix C is symmetric so there exists an orthogonal matrix $P \in M_n(\mathbb{R})$ with $P^T C P$ equal to a diagonal matrix F . By part (i), all the diagonal entries in F are positive, so there is a diagonal matrix $D \in M_n(\mathbb{R})$ such that $D^2 = F$. Then $P^T B^T B P = D^2$ whence $D^{-1} P^T B^T B P D^{-1} = I_n$. This implies that

$Q = BPD^{-1}$ is an orthogonal matrix as $Q^T Q = D^{-1} P^T B^T B P D^{-1} = I_n$ (using that D is symmetric). Rearranging, we have $B = QDP^T$, which is of the required form. 6 marks (D)

[Comments: Most of (a) with the exception of (iv) is similar to questions on problem sheets; (b)(ii) is unseen.]

(Total: 20 marks)

5. (a) State the definition of the *determinant* $\det(A)$ of an $n \times n$ matrix A over a field F . Suppose $\alpha \in F$ and $A \in M_n(F)$. Let $i \leq n$ and let $B \in M_n(F)$ be the matrix which results from multiplying row i of A by α (and leaving all the other rows the same). Using your definition, prove that $\det(B) = \alpha \det(A)$.

(6 marks)

- (b) Compute $\det(X^2 Y^3 X^T Z)$ where $X, Y, Z \in M_3(\mathbb{R})$ are given by:

$$X = \begin{pmatrix} 1 & 1 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad Z = \begin{pmatrix} 0 & 1 & 4 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

(3 marks)

- (c) Suppose H is a subgroup of the group $(G, \cdot)$.

- (i) Define what is meant by a *left coset* of H in G .

(2 marks)

Suppose that X, Y are left cosets of H in G .

- (ii) Prove that either $X \cap Y = \emptyset$ or $X = Y$.

(3 marks)

- (iii) Prove that there is a bijection $X \rightarrow Y$.

(2 marks)

- (iv) Suppose that K is also a subgroup of G and Z is a left coset of K in G . Prove that either $X \cap Z = \emptyset$ or $X \cap Z$ is a left coset of $H \cap K$ in G . Give examples to show that both of these possibilities can occur.

(4 marks)

- (a) If $A \in M_n(F)$, denote the entries of A by a_{ij} and let A_{ij} denote the ij minor, obtained by deleting row i and column j from A . The definition of $\det A$ is by induction on n . The base case $n = 1$ is just $\det A = a_{11}$. Suppose we have defined $\det$ for $n - 1 \times n - 1$ matrices. Then $\det(A) = \sum_{j=1}^n (-1)^{j+1} a_{1j} \det(A_{1j})$. The proof of the result is by induction on n , the base case $n = 1$ being trivial. Suppose the result for $n - 1 \times n - 1$ matrices. If $i = 1$ then $A_{ij} = B_{ij}$ so

$$\det(B) = \sum_j (-1)^{1+j} \alpha a_{1j} \det(B_{1j}) = \alpha \sum_j (-1)^{1+j} a_{1j} \det(A_{1j}) = \alpha \det(A).$$

If $i > 1$ then by the induction hypothesis $\det(B_{1j}) = \alpha \det(A_{1j})$ as row $i - 1$ of B_{1j} is α times row $i - 1$ of A_{1j} and all other entries are the same. Thus

$$\det(B) = \sum_{j=1}^n (-1)^{j+1} a_{1j} \det(B_{1j}) = \alpha \sum_{j=1}^n (-1)^{j+1} a_{1j} \det(A_{1j}) = \alpha \det(A).$$

This completes the inductive step.

6 marks (A)

- (b) Using $\det(AB) = \det(A)\det(B)$ and $\det(A^T) = \det(A)$ we see that what we have to compute is $\det(X)^3 \det(Y)^3 \det(Z)$. We have $\det(X) = 2$ (it's upper triangular), $\det(Y) = -1$ and $\det(Z) = 1 - 8 = -7$. So the solution is 56.

3 marks (A)

- (c) (i) This is a subset of G of the form $gH = \{gh : h \in H\}$ for some $g \in G$.

2 marks (A)

- (ii) Suppose $k \in X \cap Y$. We show $X = kH$ and the result then follows. Let $X = gH$. So there is $h_1 \in H$ with $k = gh_1$. If $h \in H$, then $kh = g(h_1h)$. So as H is a subgroup $h_1h \in H$ and $kh \in gH$. Thus $kH \subseteq gH$. We have $g = kh_1^{-1} \in kH$. So by the same argument, $gH \subseteq kH$. Thus $kH = X$, as required. **3 marks (B)**
- (iii) If $X = gH$ and $Y = fH$ we consider the map $\alpha_X : H \rightarrow X$ given by $\alpha(h) = gh$. This is clearly surjective by definition of X . It is injective as if $gh_1 = gh_2$ then $h_1 = h_2$. So α_X is a bijection and then $\alpha_Y \circ \alpha_X^{-1} : X \rightarrow Y$ is a bijection. **2 marks (B)**
- (iv) Suppose $g \in X \cap Z$. By (ii) $X = gH$ and $Z = gK$, so it will suffice to prove that $gH \cap gK = g(H \cap K)$. The containment $\supseteq$ is clear. If $f \in gH \cap gK$ write $f = gh = gk$ where $h \in H$ and $k \in K$. Then $g^{-1}f \in H \cap K$ and so $f \in g(H \cap K)$, giving the other containment. For examples we can take anything with $H = K$ and $H \neq G$.

4 marks (C)

[Comments: (a) is bookwork; (b) not seen in this form but uses standard results; (c)(i-iii) a standard question, (iv) is unseen.]

(Total: 20 marks)

6. (a) Suppose $(G, \cdot)$ is a group and $g \in G$.

(i) Define the *order* of g .

Suppose G is a cyclic group of order 12.

(ii) Give an example of such a group G .

(iii) Determine the possible orders of elements of G and say how many elements of each order there are in G .

(iv) Find subgroups A, B of G such that $A, B \neq G$, $A \cap B = \{e\}$ and $G = \{ab : a \in A, b \in B\}$. Justify your answer.

(9 marks)

(b) Suppose G, H are groups and $\phi : G \rightarrow H$ is a homomorphism.

(i) Define the term *homomorphism* and say what is meant by the *kernel*, $\ker(\phi)$, of ϕ .

(2 marks)

(ii) Prove that $N = \ker(\phi)$ is a subgroup of G and for all $g_1, g_2 \in G$ we have:

$$\phi(g_1) = \phi(g_2) \Leftrightarrow g_1 N = g_2 N.$$

(4 marks)

(c) For each of the following pairs of groups G, H , find a *non-trivial* homomorphism $\phi : G \rightarrow H$ (non-trivial here means that there is $g \in G$ with $\phi(g) \neq e_H$ and, if $G = H$, then ϕ is not the identity map.).

(i) $G = H = GL_2(\mathbb{R})$;

(ii) $G = (\mathbb{R}, +)$ and $H = (\mathbb{C}^\times, \cdot)$;

(iii) $G = GL_2(\mathbb{R})$ and $H = (\mathbb{R}^\times, \cdot)$;

(iv) $G = (\mathbb{R}, +)$ and $H = GL_2(\mathbb{R})$.

(5 marks)

(a) (i) This is the smallest $n \in \mathbb{N}$ with $g^n = e$, if such an n exists; otherwise we say that g is of infinite order. 2 marks (A)

(ii) For example, take $G = \langle \zeta \rangle \leq (\mathbb{C}^\times, \cdot)$ where $\zeta = \exp(2\pi i/12)$. 1 mark (A)

(iii) The elements of G are ζ^k for $k = 0, 1, \dots, 11$ and (for $k \neq 0$) the order of ζ^k is $12/\gcd(k, 12)$. Thus the possible orders are:

1 (for $k = 0$)

2 (for $k = 6$)

3 (for $k = 4, 8$)

4 (for $k = 3, 9$)

6 (for $k = 2, 10$)

12 (for $k = 1, 5, 7, 11$). (Other ways of solving this are possible.) 3 marks (B)

(iv) Consider A a subgroup of order 3 and B a subgroup of order 4. As $A \cap B$ is a subgroup of A and B , by Lagrange's Theorem it has order dividing 3

and 4, and so is the trivial group. It suffices to show that if $a_1, a_2 \in A$ and $b_1, b_2 \in B$ and $a_1b_1 = a_2b_2$, then $a_1 = a_2$ and $b_1 = b_2$. For then $|AB| = 12$ and so $AB = G$. But $a_1b_1 = a_2b_2$ implies $a_2^{-1}a_1 = b_2b_1^{-1} \in A \cap B$. Thus these are equal to e and we obtain $a_1 = a_2$ and $b_1 = b_2$. **3 marks (D)**

- (b) (i) This means that for all $g_1, g_2 \in G$ we have $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$. We define $\ker\phi = \{g \in G : \phi(g) = e_H\}$. **2 marks (A)**
- (ii) Use the test for being a subgroup. N is non-empty as $e_G \in N$. If $g_1, g_2 \in N$ then $\phi(g_1g_2) = e_H e_H = e_H$, so N is closed under the operation in G . Finally $e_H = \phi(g)\phi(g^{-1})$ so $\phi(g)^{-1} = \phi(g^{-1})$. Thus if $g \in N$, we have $g^{-1} \in N$. For the second part, note that

$$\phi(g_1) = \phi(g_2) \Leftrightarrow \phi(g_1^{-1}g_2) = e_H \Leftrightarrow g_1^{-1}g_2 \in N \Leftrightarrow g_1N \Leftrightarrow g_2N.$$

4 marks (1A + 3C)

- (c) (i) For example take $\phi(g) = h^{-1}gh$ for $h = \text{diag}(1, -1)$ (any invertible matrix which is not a multiple of I_2 will work here).
- (ii) Let $\phi(x) = \exp(x)$.
- (iii) Let $\phi(g) = \det(g)$.
- (iv) Let $\phi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

5 marks (2B+3D)

[Comments: (a)(i-iii) is bookwork or similar to examples on problem sheets; (iv) is unseen; (b) is bookwork from the end of the module; (c) is a mixture of seen and unseen examples.]

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once, for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH40003	1	Parts (a) - (e) of this question were generally done well. In (b) some students just wrote $AA^T=Id$ without saying that A was a matrix. In (c) some students didn't mention that they were using linearity. Part (f) was very varied, many students did not explain their steps well. For example: • Particularly few people explained why you could let $a=\sin \theta$ (i.e. $-1 < a < 1$). • Many people went from orthogonal matrix to orthnormal columns with no explanation • Many people went from orthonormal columns to matrix of the form $((a,b),(-b,a))$ with no explanation. • Many people missed cases where $a=0$ Few people attempted part (e) those who did tended to miss one of the matrices.	

MATH40003	2	Parts (a) and (c)(i) were definitional and almost all students got them without any issues. I was slightly surprised by the difficulties had with parts (b) and (c)(ii) – both questions should have been a relatively straight forward application of techniques covered in lectures. Many students made arithmetical mistakes, and many did not get all the cases for part (b). In part (c)(ii) many students did not say that the rows were linearly independent, and some made the mistake of stating that if a system has no solutions its augmented matrix must have rank 0 (a worrying misunderstanding!). Few people attempted (d). Quite a few that attempted it managed to get most of the proof assuming that there was a solution to start. I don't think anyone managed to prove that if was ok to reduce this to a question about matrices in echelon form, and there were some vague attempts at explaining why there is a solution in this case – but no fully correct solution.	
MATH40003	3	This was an easy question generally well-done by the students	
MATH40003	4	(a): Most got (i). There was some confusion of students believing that $(A-aI)(A-bI)=0$ implies $A=aI$ or $A=bI$. Part (ii) was one of the hardest: most students got that E_1 and E_2 do not intersect, but could not show that their sum is R^n. Part (iii) was one of the easiest. Part (iv) had most students incorrectly assuming v to be an eigenvector. Finally, on part (v), few students found the correct three diagonal matrices, sometimes getting an answer contradicting part (i). Part (b).(i) was one of the easiest parts, except most students did not explain properly why Bv is nonzero. Part (b).(ii) (deriving the singular value decomposition for invertible matrices, which was not seen in lectures) was definitely the hardest on the exam. Very few students solved it.	
MATH40003	5	Question was generally well answered. Several student forgot the case $n = 1$ in the def of determinant and subsequent proof in Part a - Only a few students gave a correct answer to Part C iv	

MATH40003	6	In general people did well here. A common error was in part a) iv, where many people gave subsets satisfying the conditions that were not subgroups. For part c) i, there were many possible answers; the most commonly given were inverse transpose and change of basis. Note that the inverse map and the transpose map are not homomorphisms (they reverse the order of multiplication instead of preserving it).	
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