

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Calculus and Applications

Date: Monday, May 12, 2025

Time: Start time 14:00 – End time 17:00 (BST)

Time Allowed: 3 hours

This paper has 6 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Consider the improper integral $I = \int_0^\infty f(x)dx$, where $|f(x)| \rightarrow \infty$ as $x \rightarrow 0+$ and $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Define what we mean when we say that the integral I exists. (2 marks)
- (b) (i) Take $f(x) = \frac{\log x}{1+x^2}$. Find $\lim_{x \rightarrow 0+} f(x)$, and $\lim_{x \rightarrow \infty} f(x)$, and all zeros of $f(x)$. (4 marks)
- (ii) Show that $f(\xi)$ is a unique global maximum of $f(x)$ where ξ satisfies an equation you need to find but do not need to solve. [Suggestion: Do not attempt to use the second derivative test] (6 marks)
- (iii) Sketch the graph of $f(x)$. (3 marks)
- (c) Consider the improper integral $J = \int_0^\infty \frac{\log x}{1+x^2} dx$.
- (i) Show that J exists. (3 marks)
- (ii) Calculate J . [Hint: Consider the substitution $x = 1/t$.] (2 marks)

(Total: 20 marks)

2. (a) Determine whether the following infinite series are convergent or divergent. (6 marks)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=2}^{\infty} \frac{\log(\log n)}{n^q}, \quad q \geq 1, \quad \sum_{n=0}^{\infty} \frac{e^{\sqrt{n}}}{n!}.$$

- (b) Find the power series of $\log(1+t)$ about $t = 0$, and give the values of t for which it converges. Find a rational approximation of $\int_1^{1.1} \frac{\log x}{x} dx$ that is correct to 4 decimals. (5 marks)
- (c) Let $F_n(x) = \sin(\sin(\sin(\cdots \sin(x))) \cdots)$, where the sine function appears exactly n times with n a positive integer.
- (i) Show that $\frac{dF_n}{dx} = \cos(F_{n-1}) \frac{dF_{n-1}}{dx}$, and hence

$$\frac{dF_n}{dx} = \cos(x) \prod_{m=1}^{n-1} \cos(F_m).$$

Find all zeros of $F_n(x)$ and the x -coordinates of all local maxima/minima of $F_n(x)$. [Recall the definition $\prod_{m=1}^n \alpha_m = \alpha_1 \alpha_2 \cdots \alpha_n$.] (4 marks)

- (ii) Show that $F_n(x) < \sin x < x$ for all $0 < x < \pi/2$. (3 marks)
- (iii) Determine $\lim_{n \rightarrow \infty} F_n(\pi/2)$. [Hint: Show that this is equivalent to letting $n \rightarrow \infty$ in the iteration $F_{n+1} = \sin F_n$, where $F_0 = \pi/2$.] (2 marks)

(Total: 20 marks)

3. (a) (i) Write down the first 4 terms of the power series of $1/\sqrt{1-x^2}$ about $x = 0$, and state for what values of x the series converges. (3 marks)
- (ii) Calculate $\frac{d}{dx} \sin^{-1} x$ and use the results of part 3(a)(i) above to find the first four non-zero terms of the Maclaurin series of $\sin^{-1} x$, stating the values of x for which it converges (I am using $\sin^{-1} x$ but could have used $\arcsin x$, they are the same). (4 marks)
- (iii) Find a rational approximation of $\sin^{-1}(1/10)$ that is accurate to six decimals. (3 marks)
- (b) (i) The functions $f_e(x)$ and $f_s(x)$ are periodic with period 2π , with $f_e(x)$ even and $f_s(x)$ odd. Write down their Fourier series and give formulas for their Fourier coefficients. (2 marks)

(ii) Let

$$g(x) = \begin{cases} 0 & -\pi \leq x \leq -\frac{\pi}{2} \\ \frac{\pi}{2} - |x| & |x| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

Sketch $g(x)$ and find its Fourier series.

(5 marks)

(iii) Now consider the 2π -periodic function

$$h(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ -x & 0 \leq x \leq \frac{\pi}{2} \\ x - \pi & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

Compare the graphs of $g(x)$ and $h(x)$ and thus use the results from the previous question to determine the Fourier series for $h(x)$. [You do not need to compute the Fourier coefficients directly.] (3 marks)

(Total: 20 marks)

4. (a) (i) Obtain the Fourier transform of

$$f(x) = \frac{1}{1+x^2}.$$

(3 marks)

- (ii) Give the Fourier transform of $\cos(\omega_0 x)$ in terms of the Dirac delta-functions and use convolution theorem to obtain the Fourier transform of $g(x)$ defined as the following

$$g(x) = \int_{-\infty}^{\infty} \cos(\omega_0(x-u)) e^{-|u|} du.$$

(4 marks)

- (iii) Use inverse Fourier transform of your result in (ii) to obtain an explicit expression for $g(x)$. (2 marks)

- (b) Consider the following second order linear ODE of the Euler-Cauchy type ($x > 0$):

$$x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = 4 \ln x + 2x^2 \quad (1)$$

- (i) Find the general solution of the associated *homogeneous* problem in terms of the functions of an appropriate functional basis using the change of independent variable $x = e^z$. Prove that the functions used in the solution are linearly independent. (6 marks)
- (ii) Find a particular integral and hence the general solution of the full problem given by Eq. (1). (5 marks)

(Total: 20 marks)

5. (a) Consider the following system of coupled linear differential equations:

$$\begin{aligned}\frac{dx}{dt} &= 5x + 3y \\ \frac{dy}{dt} &= -3x - y\end{aligned}$$

- (i) Find the general solution of the system in terms of the eigenvectors and eigenvalues. (4 marks)
 - (ii) Describe the asymptotic behavior of the system for large t , obtain the vector field on the y -axis and sketch the phase portrait for this system in the phase plane. (4 marks)
 - (iii) Obtain an (implicit) equation for the family of solutions for this system. (3 marks)
- (b) Consider the following first order nonlinear ODE:

$$\frac{dy}{dt} = r^2 - ry + ry^2 - y^3 \quad (2)$$

where $r \in \mathbb{R}$ is a parameter.

- (i) Find the fixed point(s), y^* , of the system and determine their stability for representative values of r . (5 marks)
- (ii) Sketch the possible values of y^* against r , and find any value of r , where there is a bifurcation and classify such bifurcations. (4 marks)

(Total: 20 marks)

6. (a) Consider the transformation:

$$\begin{aligned}x &= u^2 - v^2 \\ y &= 2uv\end{aligned}$$

- (i) Compute the Jacobian of this transformation. (4 marks)
 - (ii) Prove that $\left(\frac{\partial u}{\partial y}\right)_x = -\left(\frac{\partial v}{\partial x}\right)_y$. (4 marks)
- (b) Consider the function

$$u(x, y) = (x^2 + y^2 - 5)(y - 1).$$

- (i) Find the set of points (x_*, y_*) that form the zero contour, i.e., $u(x_*, y_*) = 0$. (2 marks)
- (ii) Find the stationary points of $u(x, y)$ and classify them stating clearly the conditions used for the classification. You should use the Hessian of $u(x, y)$ at the stationary points to classify them. Note, you do not need to classify any stationary point for which the Hessian is zero. (6 marks)
- (iii) Sketch the contour plot of $u(x, y)$ indicating the zero contour ($u = 0$) and a few other representative contours. Indicate in your sketch the positions of the stationary points and the regions of positive and negative u . Using your sketch, argue the consistency of the classifications obtained in part (ii) for the stationary points. (4 marks)

(Total: 20 marks)

1. (a) Write $I_{\epsilon, M} = \int_{\epsilon}^{x_0} f(x)dx + \int_{x_0}^M f(x)dx$. Then I exists if $\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{x_0} f(x)dx$ and $\lim_{M \rightarrow \infty} \int_{x_0}^M f(x)dx$, both exist.

seen ↓

2, A

- (b) (i) $\lim_{x \rightarrow 0} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = 0$ (x^2 beats $\log x$).
The only zeros of $f(x)$ satisfy $\log x = 0$, hence $x = 1$.

sim. seen ↓

2, A

2, A

- (ii) Differentiate to find $f'(x) = \frac{1}{x(1+x^2)} - \frac{2x \log x}{(1+x^2)^2}$. Critical points are at $x = \xi$ where $F(\xi) := 1 + \xi^2 - 2\xi^2 \log \xi = 0$, for $\xi > 0$. Need to show that $F(\xi)$ has one root only, i.e. a unique critical point. We will also need $F'(\xi) = -4\xi \log \xi$. First, $F(0) = 1$ and $F(1) = 2$, and in fact $1 \leq F(\xi) \leq 2$ for $0 \leq \xi \leq 2$, the function $F(\xi)$ is monotonic increasing in $0 < \xi < 1$, since $F'(\xi) > 0$ in this interval.

unseen ↓

sim. seen ↓

2, B

Now consider the interval $1 < \xi < \infty$, clearly $F'(\xi) < 0$ and hence $F(\xi)$ is monotonic decreasing here. Since $F(1) = 2$ and $F(\infty) = -\infty$, there must be a unique root, and so a unique critical point.

2, A

We know from part 1(a)i that $f(0) = -\infty$, $f(\infty) = 0$ and $x = 1$ is the only zero of f . By continuity the critical point has to be a maximum.

2, A

- (iii) A sketch with the necessary labels is given in figure 1.

3, B

- (c) (i) Write $J = \int_0^1 \frac{\log x}{1+x^2} dx + \int_1^\infty \frac{\log x}{1+x^2} dx$, and let $J_\epsilon = \int_\epsilon^1 \frac{\log x}{1+x^2} dx$, and $J_M = \int_1^M \frac{\log x}{1+x^2} dx$.

unseen ↓

Estimate, $0 \leq -J_\epsilon = \int_\epsilon^1 \frac{-\log x}{1+x^2} dx < \int_\epsilon^1 (-\log x) dx = [-x \log x]_\epsilon^1 + (1 - \epsilon)$, and hence $\lim_{\epsilon \rightarrow 0} J_\epsilon$ is bounded as $\epsilon \rightarrow 0$.

1, B

At the other extreme, $J_M < \int_1^M \frac{\log x}{x^2} dx$ since $x \geq 1$, and this can be integrated exactly by parts to give $J_M < \left[-\frac{\log x}{x}\right]_1^M + \int_1^M \frac{1}{x^2} dx = -\frac{\log M}{M} + 1 - \frac{1}{M}$, and hence $\lim_{M \rightarrow \infty} J_M$ exists.

2, C

- (ii) Since J exists, we can make the substitution $x = 1/t$ as suggested, i.e. $dx = (-1/t^2)dt$ etc.. Calculating we find $J = \int_0^\infty \frac{\log x}{1+x^2} dx = \int_\infty^0 \frac{-\log t (-1/t^2)}{1+1/t^2} dt = -\int_0^\infty \frac{\log t}{1+t^2} dx = -J$. Hence $J = -J$ and since the integral exists we have $J = 0$.

2, D

2. (a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$, by the integral test $\int_1^{\infty} \frac{dx}{x^2} < \infty$, converges.

sim. seen $\Downarrow$

1, A

$\sum_{n=2}^{\infty} \frac{\log(\log n)}{n^q}$. Integral test and integrate by parts and consider $M \rightarrow \infty$ at the end:

$$\begin{aligned} \int_2^M \frac{\log \log x}{x^q} dx &= \left[\frac{x^{1-q}}{1-q} \log \log x \right]_2^M - \frac{1}{1-q} \int_2^M x^{1-q} \frac{1}{x \log x} dx \\ &= \left[\frac{x^{1-q}}{1-q} \log \log x \right]_2^M - \frac{1}{1-q} \int_2^M \frac{1}{x^q \log x} dx \end{aligned}$$

Clearly $q > 1$ gives a convergent integral; x to any power beats $\log$, or formal use of L'Hopital, second term $\int_2^M \frac{1}{x^q \log x} dx < \text{const.} + \int_e^M \frac{1}{x^q} dx < \infty$.

Remains to discuss $q = 1$. Now $\int_2^M \frac{\log \log x}{x} dx = [\log x \log \log x]_2^M - \log M$, which is unbounded.

Hence, convergence only if $q > 1$.

3, B

$\sum_{n=0}^{\infty} \frac{e^{\sqrt{n}}}{n!}$. Ratio test,

$$\begin{aligned} \frac{e^{\sqrt{n+1}}}{(n+1)!} \frac{n!}{e^{\sqrt{n}}} &= \frac{e^{\sqrt{n+1}-\sqrt{n}}}{n+1} = \frac{e^{\sqrt{n}(\sqrt{1+1/n}-1)}}{n+1} = \frac{\exp\left(\sqrt{n} \frac{1+\frac{1}{n}-1}{\sqrt{1+1/n}+1}\right)}{n+1} \\ &< \frac{e^{-\sqrt{n}}}{n+1} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Series converges.

2, C

seen $\Downarrow$

- (b) $\log(1+t) = t - t^2/2 + t^3/3 - t^4/4 + \dots$, and the series converges for $|t| < 1$.

2, A

Put $x = 1+t$ to find $\int_1^{1.1} \frac{\log x}{x} dx = \int_0^{0.1} \frac{\log(1+t)}{1+t} dt$, and by the result above along with the binomial theorem we have

sim. seen $\Downarrow$

$$\begin{aligned} \frac{\log(1+t)}{1+t} &= (t - t^2/2 + t^3/3 - t^4/4 + \dots) (1 - t + t^2 - t^3 + \dots) \\ &= t - \frac{3}{2}t^2 + (1 + 1/2 + 1/3)t^3 - (1 + 1/2 + 1/3 + 1/4)t^4 + \mathcal{O}(t^5) \Rightarrow \\ &\int_0^{0.1} \frac{\log(1+t)}{1+t} dt = \frac{1}{2}0.1^2 - \frac{1}{2}0.1^3 + \frac{11}{24}0.1^4 - \dots \end{aligned}$$

The term $\frac{11}{24}0.1^4 < 0.5 \times 10^{-4}$, hence the value $\frac{1}{2}0.1^2 - \frac{1}{2}0.1^3 = \frac{1}{200} - \frac{1}{2000}$ is a rational approximation with four decimals accuracy.

3, D

- (c) (i) Use the chain rule, $dF_n/dx = \cos(F_{n-1})dF_{n-1}/dx$, and proceed iteratively

unseen $\Downarrow$

$$\begin{aligned} \frac{dF_n}{dx} &= \cos(F_{n-1}) \cos(F_{n-2}) \frac{dF_{n-2}}{dx} = \dots = \cos(F_{n-1}) \cos(F_{n-2}) \dots \cos(\sin(x)) \cos(x) \\ &= \cos(F_{n-1}) \cos(F_{n-2}) \dots \cos(F_1) \cos(x), \end{aligned}$$

as required.

2, D

The zeros are the same as the zeros of $\sin(x)$, namely $\pm n\pi$, $n = 0, 1, 2, \dots$. For the local maxima/minima, since $|F_m| \leq 1$ we cannot have $\cos(F_m) = 0$; the only zeros are when $\cos(x) = 0$, i.e. $x = \pm(2n+1)\pi/2$, $n = 0, 1, 2, \dots$.

2, C

- (ii) Let $g(x) = x - \sin x$, then $g'(x) = 1 - \cos x > 0$ for $0 < x < \pi/2$, hence g is increasing from 0 and $\sin x < x$. Next, let $h(x) = \sin x - F_n(x)$ and differentiate to find $h'(x) = \cos x \left(1 - \prod_{m=1}^{n-1} \cos F_m\right)$; in $0 < x < \pi/2$ we deduce that $g'(x) > 0$, hence the function is monotonic increasing from 0 and $F_n < \sin x$.

seen ↓

sim. seen ↓

1, A

2, B

- (iii) $F_0 = \pi/2$, $F_1 = \sin(\pi/2) = \sin F_0$ and by repeatedly taking the sine of both sides we have $F_{n+1} = \sin F_n$. Suppose that $\lim_{n \rightarrow \infty} F_n = F^*$. Then we must have $F^* = \sin F^*$, and the only solution of this equation is $F^* = 0$.

unseen ↓

2, D

3. (a) (i) Binomial theorem:

sim. seen ↓

$$\begin{aligned}(1 - x^2)^{-1/2} &= 1 + \frac{1}{2}x^2 + \left(\frac{-1}{2}\right) \left(\frac{-3}{2}\right) \frac{(-x^2)^2}{2!} + \left(\frac{-1}{2}\right) \left(\frac{-3}{2}\right) \left(\frac{-5}{2}\right) \frac{(-x^2)^3}{3!} \dots \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots\end{aligned}$$

The series converges for $x^2 < 1$, i.e. $-1 < x < 1$.

3, A

sim. seen ↓

- (ii) Let $y = \sin^{-1} x$, hence $\sin y = x$; implicit differentiation gives $\cos y \, dy/dx = 1$, hence $dy/dx = 1/\cos y = 1/\sqrt{1 - \sin^2 y} = 1/\sqrt{1 - x^2}$. Use the result in part (i) above and integrate between 0 and x to find

4, B

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots, \quad -1 < x < 1.$$

- (iii) If $x = 0.1$, the last term can be dropped since it is smaller than 10^{-7} . The term $\frac{3}{40}0.1^5$ must be kept. Hence the rational approximation is $0.1 + \frac{10^{-3}}{6} + \frac{3 \times 10^{-6}}{4}$.

sim. seen ↓

3, C

- (b) (i)

seen ↓

$$\begin{aligned}f_e(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), & a_n &= \frac{2}{\pi} \int_0^{\pi} f_e(x) \cos(nx) \, dx, \\ f_s(x) &= \sum_{n=1}^{\infty} b_n \sin(nx), & b_n &= \frac{2}{\pi} \int_0^{\pi} f_s(x) \sin(nx) \, dx\end{aligned}$$

2, A

(ii) A sketch is given in figure 2.

seen ↓

2, A

The function is even, hence the Fourier series is as for f_e above with

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\pi}{2} - x \right) dx = \frac{2}{\pi} \left[-(\pi/2 - x)^2 \frac{1}{2} \right]_0^{\pi/2} = \frac{\pi}{4}.$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\pi}{2} - x \right) \cos(nx) dx = \frac{2}{\pi} \left[\left(\frac{\pi}{2} - x \right) \frac{\sin nx}{n} \right]_0^{\pi/2} + \frac{2}{\pi} \int_0^{\pi/2} \frac{\sin nx}{n} dx \\ &= \frac{2}{\pi} \left[-\frac{\cos nx}{n^2} \right]_0^{\pi/2} = \frac{2}{\pi n^2} \left(1 - \cos \frac{n\pi}{2} \right) \end{aligned}$$

3, A

unseen ↓

(iii) Observe that $h(x)$ is constructed from $g(x)$ by a shift of $+\pi/2$ and a reflection about the x -axis. Hence

$$\begin{aligned} h(x) &= -g(x - \pi/2) = -\frac{a_0}{2} - \sum_{n=1}^{\infty} a_n \cos(n(x - \pi/2)) \\ &= -\frac{a_0}{2} - \sum_{n=1}^{\infty} a_n [(\cos nx \cos(n\pi/2) + \sin nx \sin(n\pi/2))] \\ &= -\frac{a_0}{2} - \sum_{n=1}^{\infty} [a_n \cos(n\pi/2) \cos nx + a_n \sin(n\pi/2) \sin nx], \end{aligned}$$

with a_0, a_n as calculated in part b(ii) above.

3, D

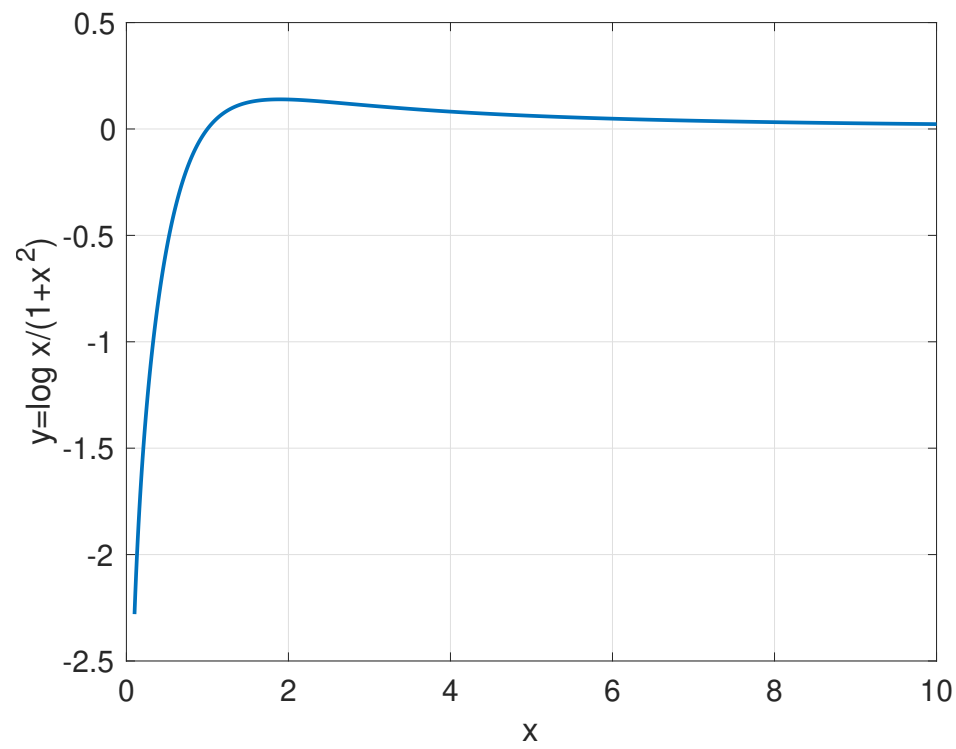


Figure 1: Figure for problem 1

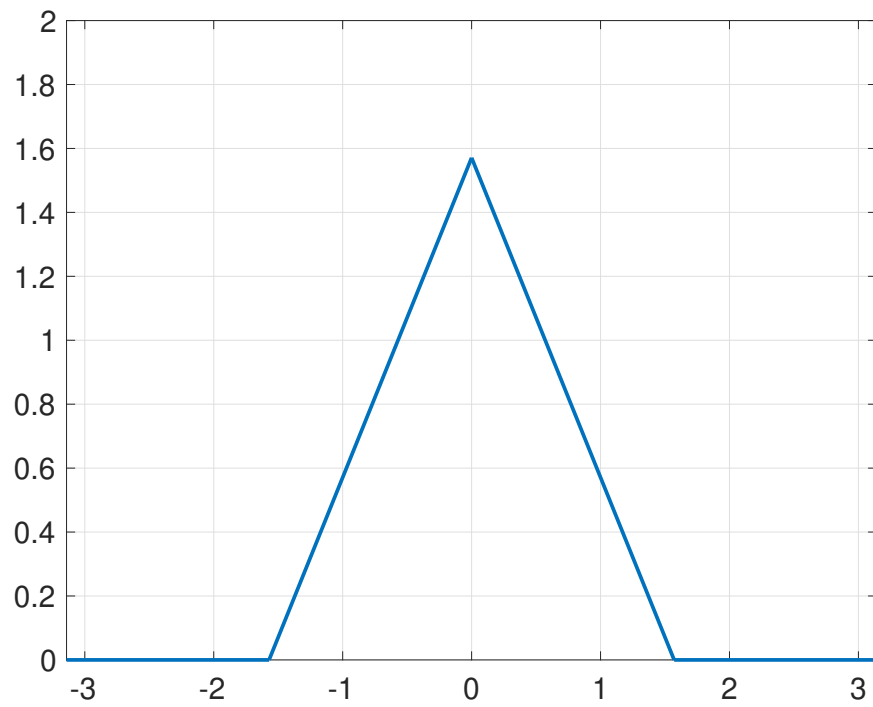


Figure 2: Figure for problem 3(b)ii

Review of mark distribution:

Total A marks: 24 of 32 marks

Total B marks: 15 of 20 marks

Total C marks: 9 of 12 marks

Total D marks: 12 of 16 marks

Total marks: 60 of 80 marks

Total Mastery marks: 0 of 20 marks

- 4.(a(i)) Consider $h(x) = e^{-|x|}$. We have (from the examples in the lectures, or can be obtained using the definition of the Fourier transform):

meth seen ↓

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x} dx = \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}.$$

So, we observe $f(x) = 1/2\hat{h}(x)$ and using symmetry formula we have

$$\mathcal{F}\left\{\frac{1}{2}\hat{h}(x)\right\} = \pi h(-\omega) = \pi e^{-|\omega|}.$$

- (a(ii)) We have from lectures (or can be obtained from definitions):

3, B

$$\mathcal{F}\{\cos(\omega_0 x)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0).$$

unseen ↓

We observe that $g(x) = \cos(\omega x) * h(x)$. From last part's result and using the convolution theorem we have

2, A

$$\mathcal{F}\{\cos(\omega x) * h(x)\} = \frac{2\pi}{(1+\omega^2)} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

From problem sheet 1 or using properties of Dirac delta function, we note that the prefactor in the above can be evaluated at ω_0 so we have:

$$\mathcal{F}\{g(x)\} = \frac{2\pi}{(1+\omega_0^2)} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

(Marks are not deducted if the last step is not done, this is needed though for the next part.)

2, D

- (a(iii)) Given the Fourier transform of function $g(x)$ obtained above and linear property of Fourier transforms and the fact ω_0 is a constant and noting the Fourier transform of $\cos(\omega_0 x)$, (or using inverse Fourier formula) we obtain:

unseen ↓

$$g(x) = \frac{2}{(1+\omega_0^2)} \cos(\omega_0 x).$$

- (b(i)) We have (from the lectures, or can be obtained):

2, D

$$x \frac{dy}{dx} = \frac{dy}{dz},$$

meth seen ↓

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}.$$

So, we obtain the following ODE in terms of $y(z)$:

$$\frac{d^2 y}{dz^2} - 4 \frac{dy}{dz} + 4y = 4z + 2e^2 z.$$

3, A

The characteristic equation of the corresponding homogeneous ODE is:

$$\lambda^2 - 4\lambda + 4 = 0,$$

which has a repeated root of $\lambda = 2$. So, we have from the lectures the following two functions as the basis for the solution vector space:

$$y_1 = e^{2z}, \quad y_2 = ze^{2z}.$$

So the complementary function, the general solution of the homogeneous problem is:

$$y_{CF} = c_1 e^{2z} + c_2 z e^{2z} = c_1 x^2 + c_2 x^2 \ln x$$

.

1, A

We calculate the Wronskian and check if it is non-zero to prove linear independence:

$$W = e^{4z} + 2ze^{4z} - 2ze^{4z} = e^{4z} \neq 0.$$

2, B

(b(ii)) We will stay with the ODE in terms of independent variable z , we try the ansatz $y_{PI}(z) = az + b + cz^2 e^{2z}$, where a , b and c are unknown constants to be determined. By plugging the ansatz in the equation and some algebra, we obtain $a = 1$, $b = 1$ and $c = 1$.

meth seen ↓

a and b part

2, A

c part

2, C

$$y_{GS} = y_{CF} + y_{PI} = c_1 e^{2z} + c_2 z e^{2z} + z + 1 + z^2 e^{2z} = c_1 x^2 + c_2 x^2 \ln x + \ln x + 1 + (\ln x)^2 x^2$$

.

1, C

5.(a(i)) We have for matrix A :

meth seen ↓

$$A = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}.$$

We calculate eigenvalues and eigenvectors of the matrix A :

$$\lambda = 2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

This eigenvalue is repeated and only associated to one eigenvector, so the matrix is not diagonalisable and we need to use Jordan normal form. We look for

$$W^{-1}AW = J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

First column of W is $\vec{v}_1$ and we look for a vector $\vec{w}_2$ as the second column of W to satisfy the equation above. We obtain:

$$3w_{2x} + 3w_{2y} = 1 \Rightarrow \vec{w}_2 = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}.$$

Therefore, from the lecture or can be derived using the Jordan normal form, we have for the general solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (c_1 e^{2t} + c_2 t e^{2t}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}.$$

4, A

(a(ii)) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an unstable fixed point and the solutions asymptotically approach infinity along $\vec{v}_1$. The vector field on the y axis can be calculated by:

meth seen ↓

$$A \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ -y \end{pmatrix}.$$

Figure 1 shows the phase portrait for this system. The solutions should be parallel to the vector field calculated on the y axis and also asymptotically become parallel to $\vec{v}_1$.

4, A

(a(iii)) We need to solve the following dimensionally homogeneous ODE:

sim. seen ↓

$$\frac{dy}{dx} = \frac{-3x - y}{5x + 3y}.$$

We let $u = y/x$ and we obtain:

$$x \frac{du}{dx} = \frac{-3 - u}{5 + 3u} - u = -3 \frac{(u + 1)^2}{5 + 3u}$$

This is a separable ODE, so we can integrate and we obtain:

$$3 \ln \left(\frac{y}{x} + 1 \right) - \frac{2x}{x + y} = -3 \ln x + c.$$

3, C

- (b(i)) The fixed points of the system are obtained by putting the right hand side to zero. We have

$$f(y) = (r - y)(r + y^2) = 0 \Rightarrow y_1^* = r, \quad y_2^* = \pm\sqrt{-r}$$

So, we have upto three fixed points. We consider two cases of $r < 0$ and $r > 0$. In Figure 2 we have determined the stability of these fixed points for the different values of r by plotting $f(y)$ against y . (note that one could also plot $r = -1$ as a separate case as in this case there are only two fixed points).

5, B

- (b(ii)) The bifurcation diagram for this system is sketched in Figure 3. There is a supercritical pitchfork bifurcation at $r = 0$ and a transcritical bifurcation at $r = -1$.

2, D

2, D

6.(a(i)) The Jacobian J of the transformation is:

seen/sim.seen ↓

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

So, given the form of the transformation we obtain:

$$J = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}.$$

4, B

(a(ii)) The Jacobian for inverse transformation is:

seen/sim.seen ↓

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

But, this transformation is also equal J^{-1} , so we obtain:

$$J^{-1} = \frac{1}{\det(J)} \begin{pmatrix} 2u & 2v \\ -2v & 2u \end{pmatrix} = \frac{1}{\det(4(u^2 + v^2))} \begin{pmatrix} 2u & 2v \\ -2v & 2u \end{pmatrix}.$$

We can see from this result that $\left(\frac{\partial u}{\partial y}\right)_x = -\left(\frac{\partial v}{\partial x}\right)_y$ as required.

4, D

(b(i))

seen/sim.seen ↓

$$u(x, y) = (x^2 + y^2 - 5)(y - 1) = x^2y + y^3 - 5y - x^2 - y^2 + 5$$

Zero contour ($u = 0$):

$$x_*^2 + y_*^2 = 5, \quad y_* = 1$$

2, A

(b(ii)) We need to evaluate the gradient of u and set it to zero. We have:

unseen ↓

$$\frac{\partial u}{\partial x} = 2xy - 2x = 0 \Rightarrow x = 0 \quad \text{or} \quad y = 1,$$

$$\frac{\partial u}{\partial y} = x^2 + 3y^2 - 5 - 2y = 0$$

Given the above, and by putting the solutions of the first equation into the second equation, we have the following stationary points:

$$P_1 = (0, -1), \quad P_2 = (0, \frac{5}{3}) \quad P_3 = (-2, 1) \quad P_4 = (2, 1).$$

For the character of the stationary points we calculate the Hessian. We have:

3, A

$$H(x, y) = \begin{bmatrix} 2y - 2 & 2x \\ 2x & 6y - 2 \end{bmatrix}.$$

Now, we can evaluate Hessian at each stationary point and based on the value of trace and determinant of the matrix, we can decide if the stationary point is a maximum, minimum or saddle point.

$$H_{P_1} = \begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix} \Rightarrow \tau = -12 < 0, \quad \Delta = 32 > 0 \Rightarrow P_1 \text{ is a maximum}$$

$$H_{P_2} = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & 8 \end{bmatrix} \Rightarrow \tau = \frac{27}{3} > 0, \quad \Delta = \frac{32}{3} > 0 \Rightarrow P_2 \text{ is a minimum.}$$

$$H_{P_3} = \begin{bmatrix} 0 & -4 \\ -4 & 4 \end{bmatrix} \Rightarrow \tau = 4, \quad \Delta = -16 < 0 \Rightarrow P_3 \text{ is a saddle-point.}$$

$$H_{P_4} = \begin{bmatrix} 0 & 4 \\ 4 & 4 \end{bmatrix} \Rightarrow \tau = 4, \quad \Delta = -16 < 0 \Rightarrow P_4 \text{ is a saddle-point.}$$

Given the form of τ above, for $1 < x < 4$ line $y = 0$ is a maximum. For $x < 1$ and $x > 4$ $\tau > 0$ so $y = 0$ is a minimum. Note for $x = 1$ and $x = 4$ the Hessian is zero, so we need higher order derivatives to classify such points, which is out of the scope of this course.

- (b(iii)) Figure 4 shows the sketch of the level sets of u , the stationary points and the sign of the function u .

For the P_3 and P_4 points, which based on the Hessian, we obtained to be a minimum, we see that the value of the function u in the vicinity of the point becomes negative (decreases from 0) in certain directions and it becomes positive (increases from 0) in other directions, so this is consistent with these points being a saddle-point. We also have $u(P_1) < 0$, so this is also consistent with P_1 being a minimum. Finally, we note that $u(P_2) > 0$, so this is consistent with this point being a maximum.

3, A

unseen ↓

3, C

1, B

Review of mark distribution:

Total A marks: 24 of 32 marks

Total B marks: 15 of 20 marks

Total C marks: 9 of 12 marks

Total D marks: 12 of 16 marks

Total marks: 60 of 80 marks

Total Mastery marks: 0 of 20 marks

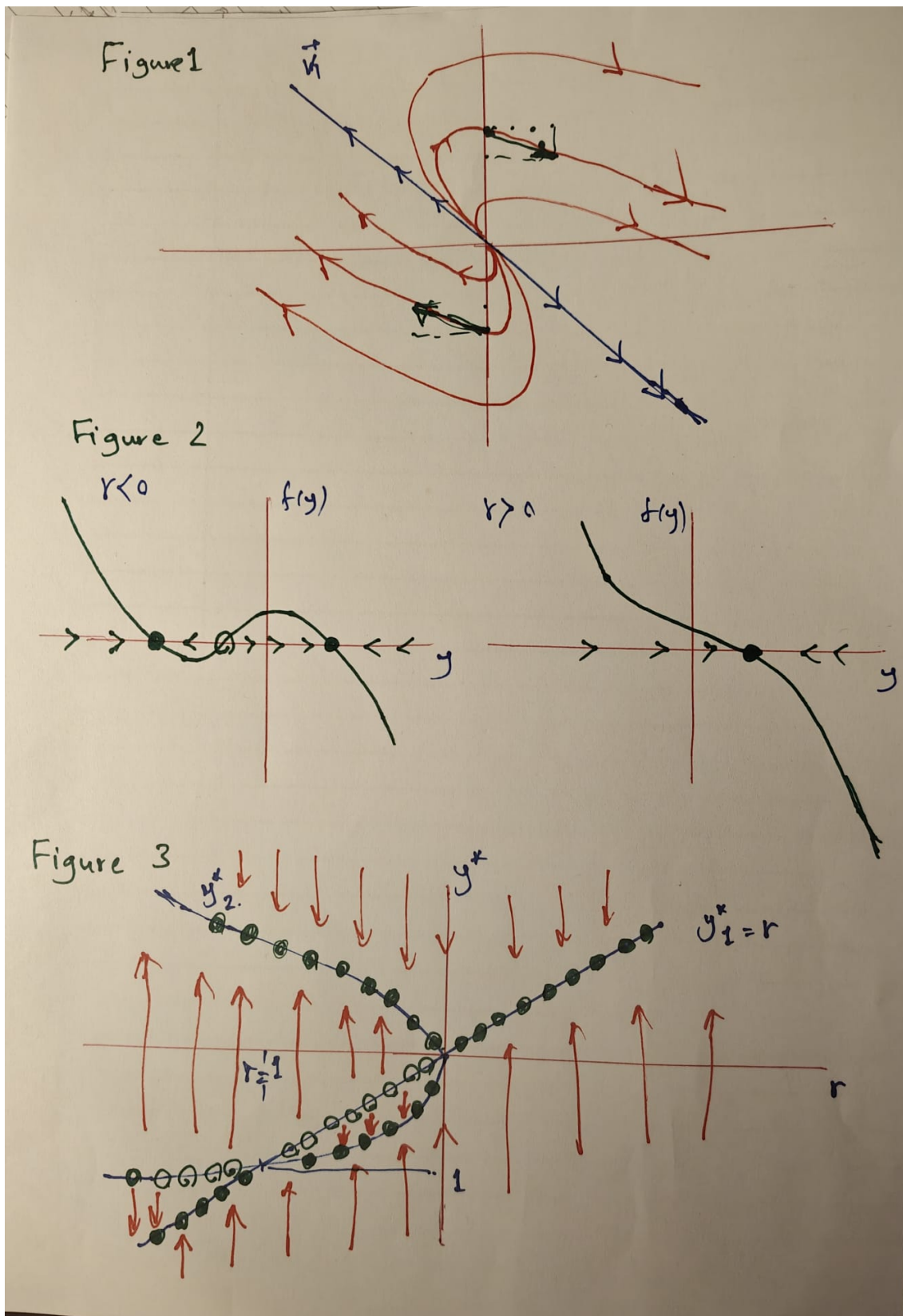


Figure 1: Figures 1-3 related to problem 5

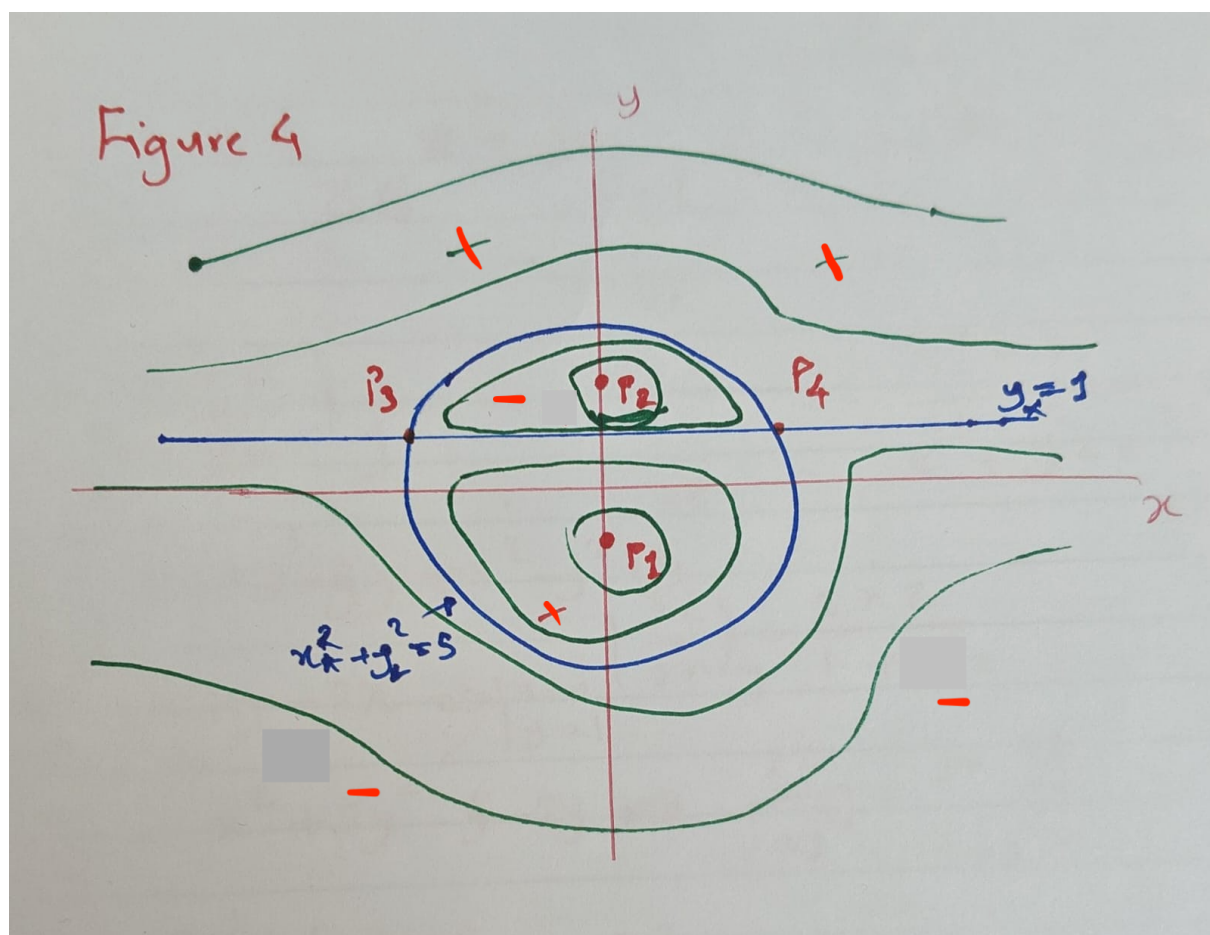


Figure 2: Figures 4 related to problem 6b

MATH40004 Calculus and Applications Markers comments

- Question 1 Question was answered well. A lot of people missed how to apply monotonicity arguments to show that the global maximum is unique. The use of limits in calculating improper integrals was very important in this question and not everyone was comfortable with the methods.
- Question 2 The question was answered fairly well. The main obstacle seemed to be the use of Taylor expansions about a point different from zero. Also, the binomial theorem was not used where needed by many people.
- Question 3 a) i) Many students opted to derive the Maclaurin series in terms of derivatives of f , which led to success rarely. Those who used the binomial theorem were more successful. A fair number of students also forgot to put the domain of convergence.
a) ii) Answered well, with almost all students correctly integrating their power series from i). However, it was again common to omit the domain of convergence.
a) iii) Most students knew to substitute 0.1 into their series from ii), but many did not correctly justify their answer, it being commonly written that an error of $<10^{-6}$ was sufficient.
b) i) Answered well for the most part. Most common errors were forgetting factors in front the integrals and incorrect limits on the integrals.
b) ii) Almost all students drew the correct graph, and many students correctly integrated for a_n . However, a significant number made a mistake at the final moment claiming that $\cos(n\pi/2)$ is always zero.
b) iii) A large number of students did not expand $\cos(n(x-\pi/2))$, and so did not give a true Fourier series as their final answer. Despite the suggestion given in the question, almost no students left their answer in terms of a_0 and a_n , which sometimes led to more errors than necessary.
- Question 4 The performance on the question 4 was bimodal, most people overall did well in this question particularly in part b (some people did not use Wronskian for proving independence which is the method discussed in the course). For part a several people had difficulty with Fourier transform of cosine function even though it was an example in the lectures. A second smaller group of students did really poorly in this question.
- Question 5 The cohort answered the questions well. Many students were able to apply advanced concepts such as Jordan normal forms and bifurcation diagrams successfully but encountered problems when locating and naming the two bifurcations.

Question 6

On the whole this question was done well. Many students did not realise the connection between parts (a)(i) and (a)(ii), which led to convoluted answers or to mistakes. Other common errors included the definition of the Jacobian, calculating the Hessian, and the classifications thereafter. A surprising number of students confused $x = \pm 2$, $y = 1$ for the coordinates $(1, \pm 2)$ in (b)(ii), which led to confusion in (b)(iii).