

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2026

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Calculus and Applications

Date: Wednesday, 06 May, 2026

Time: Start time 14:00 – End time 16:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) State without proof Taylor's theorem for the function $f(x)$ about the point $x = a$. Give the remainder in its local form. (2 marks)
- (b) Use Taylor's theorem to expand $\log(x)$ about the point $x = 1$ (you do not need to give the remainder). Find for what values of x the series is convergent? (3 marks)
- (c) Consider the function $g(x) = \frac{\log(x)}{x^2 - 1}$, $x > 0$.
- (i) Calculate the following limits: (3 marks)

$$\lim_{x \rightarrow 0^+} g(x), \quad \lim_{x \rightarrow 1} g(x), \quad \lim_{x \rightarrow \infty} g(x)$$

- (ii) Use your results in part (b) to write $(x + 1)g(x)$ as a convergent power series near $x = 1$ and state its radius of convergence. (2 marks)
Hence express $\log(3) - \log(2)$ as a rational number that is correct to two decimals. (3 marks)
- (iii) Show that $\frac{d^n}{dx^n} [(x + 1)g(x)] = (x + 1)g^{(n)}(x) + ng^{(n-1)}(x)$, where $g^{(m)} = \frac{d^m g}{dx^m}$ for any positive integer m . Hence, or otherwise, determine how many derivatives of $g(x)$ exist at $x = 1$. (3 marks)
- (iv) Let $I = \int_0^\infty g(x)dx$ be the improper integral of $g(x)$. Determine whether I exists. (4 marks)

(Total: 20 marks)

2. (a) (i) Given $x > x_0 > 0$ and $A > 0$ calculate the integral

(2 marks)

$$\int_{x_0}^x \frac{d\xi}{(1 + A\sqrt{\xi})\sqrt{\xi}}.$$

[Hint: Use the substitution $y = \sqrt{\xi}$.]

- (ii) A cylinder of radius R is filled with water to a height h_0 . Water is lost from the cylinder in two ways: through a circular hole of radius r at the centre of its base, and, through the wetted curved side wall which is porous. The rate at which liquid is lost from the hole is equal to the hole's area times $\sqrt{gh}$ where g is the acceleration of gravity and $h(t)$ is the instantaneous height at time t . The rate at which liquid is lost through the porous wall is equal to a positive constant k times the *wetted* curved area. Show that if $V(t)$ is the volume of liquid in the cylinder after time t , then

$$\frac{dV}{dt} = -\pi r^2 \sqrt{gh} - 2\pi k R h.$$

What are the dimensions of the constant k ?

(3 marks)

- (iii) Given that $h(0) = h_0$ find h at any subsequent time $t > 0$ and show that the time t_e it takes for the tank to empty is given by

$$t_e = \frac{R \log(1 + \alpha \sqrt{h_0})}{k}, \quad \text{where} \quad \alpha = \frac{2kR}{r^2 \sqrt{g}}.$$

(4 marks)

- (iv) By considering $\lim_{k \rightarrow 0} h(t)$ in your result above, find the corresponding solution when there are no losses due to a porous curved wall, and determine the time t_s to empty the tank. Explain physically why we should have $t_e < t_s$.

(3 marks)

- (b) The function

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ 0 & -\pi < x < 0 \end{cases},$$

is extended periodically on the real line to produce a 2π -periodic function $g(x)$.

- (i) Find the Fourier series of $g(x)$. (4 marks)
- (ii) What value does your Fourier series converge to at any points of discontinuity? Confirm this for $x = 0, \pi, -121\pi$. (2 marks)
- (iii) Use your results in part (b)(i) to find an alternating infinite series containing rational terms and which converges to π . (2 marks)

(Total: 20 marks)

3. (a) (i) Consider the following system of coupled linear differential equations:

$$\begin{aligned}\frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= -4x + y\end{aligned}$$

Find the general solution of the system in terms of the eigenvectors and eigenvalues. If needed then write the general solution in terms of real functions. (5 marks)

- (ii) Describe the asymptotic behaviour of the system for large t , obtain the vector field on the x -axis and sketch the phase portrait for this system in the phase plane. (4 marks)
- (iii) Now consider the corresponding system of non-homogenous linear ODEs:

$$\begin{aligned}\frac{dx}{dt} &= x + y + e^t \\ \frac{dy}{dt} &= -4x + y + 5\end{aligned}$$

Find a particular integral using appropriate ansatz and write the general solution of this system of ODEs. (4 marks)

- (b) Consider the following nonlinear ODE:

$$\frac{dy}{dx} = -\frac{3y \log y + 3y^2}{x + xy}$$

- (i) Show that this ODE is not exact. (2 marks)
- (i) Find an integrating factor of the form $\lambda = x^m y^n$ with m and n in $\mathbb{R}$ being parameters to turn the equation into exact form and use it to obtain a solution in implicit form for the ODE. (5 marks)

(Total: 20 marks)

4. (a) Consider the following first order nonlinear ordinary differential equation:

$$\frac{dy}{dt} = ry - \frac{y}{1+y}$$

where $r \in \mathbb{R}$ is a parameter.

- (i) Find the fixed point(s), and singular point(s) of the system and determine their stability by sketching ry and $\frac{y}{1+y}$ against y for representative values of r , or otherwise. (5 marks)
- (ii) Sketch the bifurcation diagram that shows the values of fixed and singular point(s) against r , and find any value of r , where there is a bifurcation and classify such bifurcations. If $r = \frac{1}{4}$, for what values of initial condition $y(t = 0)$ does the solution asymptotically approach a stable fixed point? (4 marks)
- (b) Consider the following second order linear ODE, known as the *Airy* equation:

$$\frac{d^2y}{dx^2} - xy = 0$$

Assuming $\lim_{x \rightarrow \pm\infty} y = 0$, by taking Fourier transform of both sides of this ODE, obtain a solution for $y(x)$ in terms of an integral of a real function. (5 marks)

- (c) Write the general form of the Taylor expansion for a multi-variate function $f(x, y)$ around the point (x_0, y_0) up to order two, in terms of the gradient and Hessian of the function f . Consider $z(x, y)$ given in implicit form:

$$\sin(xz) = z + yx$$

Obtain the Taylor expansion of $z(x, y)$ up to the first order around $x = 0$ and $y = 1$.

(6 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2026

This paper is also taken for the relevant examination for the Associateship.

MATH-40004

Calculus & Applications (Solutions)

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1. (a) $f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \dots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + R_n$,
where $R_n = \frac{1}{(n+1)!}(x-a)^{n+1} f^{(n+1)}(\xi)$ with ξ between x and a .

seen ↓

2, A

(b)

sim. seen ↓

$$\begin{aligned}\log(x) &= \log(1+x-1) = \log(1) + (x-1) \left[\frac{1}{x} \right]_1 + \frac{1}{2!}(x-1)^2 \left[-\frac{1}{x^2} \right]_1 + \frac{1}{3!}(x-1)^3 \left[\frac{2!}{x^3} \right]_1 - \dots \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots\end{aligned}$$

2, A

Convergence for $-1 < x-1 \leq 1$, that is $0 < x \leq 2$.

1, B

sim. seen ↓

(c) (i)

$$\lim_{x \rightarrow 0^+} \frac{\log x}{x^2 - 1} = - \lim_{x \rightarrow 0^+} \log x = +\infty$$

1, A

$$\lim_{x \rightarrow 1} \frac{\log x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{1/x}{2x} = \frac{1}{2}$$

1, A

$$\lim_{x \rightarrow \infty} \frac{\log x}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{1/x}{2x} = 0$$

1, A

- (ii) $(x+1)g(x) = \frac{\log(x)}{x-1} = 1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^3}{4} + \dots$. Radius of convergence is $0 < x < 2$ as before.

sim. seen ↓

Notice that $\log(3) - \log(2) = \log(1.5)$ and so we have from above

unseen ↓

$$\frac{\log(3/2)}{1/2} = 1 - \frac{(1/2)}{2} + \frac{(1/2)^2}{3} - \frac{(1/2)^3}{4} + \frac{(1/2)^4}{5} + \dots$$

2, B

2, B

hence

1, D

$$\log(3/2) = \frac{1}{2} - \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} - \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5} + \dots$$

Series is alternating, so for 2 decimals accuracy we need the largest ignored term to be $< 0.005 = 1/200$, i.e. $\frac{(1/2)^n}{n} < 1/200$. Since $\frac{(1/2)^5}{5} = \frac{1}{32 \times 5} > 1/200$ but $\frac{(1/2)^6}{6} = \frac{1}{64 \times 6} < 1/200$, the appropriate rational approximation is

$$\log(3) - \log(2) \approx \frac{1}{2} - \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} - \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5}$$

unseen ↓

1, C

- (iii) Can do by induction or directly. For the former, clearly true for $n = 1$ and $n = 2$. Assume true for n , i.e. $\frac{d^n}{dx^n}[(x+1)g] = ng^{(n-1)} + (x+1)g^{(n)}$. Taking one more derivative gives

2, D

$$\frac{d^{n+1}}{dx^{n+1}}[(x+1)g] = ng^{(n)} + g^{(n)} + (x+1)g^{(n+1)} = (n+1)g^{(n)} + (x+1)g^{(n+1)}.$$

Since $(x + 1)g$ has derivatives of all orders existing at $x = 1$ (we have given the power series above), then can establish recursively that if $(x + 1)g$ has a derivative of order n at $x = 1$, then so does $g(x)$ from the formula above.

sim. seen $\Downarrow$

- (iv) $I = \lim_{\epsilon \rightarrow 0} \lim_{M \rightarrow \infty} \int_{\epsilon}^M \frac{\log x}{x^2 - 1} dx$. For the lower limit the function behaves like $\log(x)$ and since $\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{x_0} \log(x) dx$ is bounded, so no issues at $x = 0$. For the M limit we note that for large x it is enough to consider $\int_{x_0}^M \frac{\log x}{x^2} dx$ (in fact there exists x_0 s.t. $\int_{x_0}^M \frac{\log x}{x^2 - 1} dx < 2 \int_{x_0}^M \frac{\log x}{x^2} dx$, so enough to study the latter integrand - students may not state such details; credit given if they identify the right integrand for $x \gg 1$ without detailed analysis). Hence we integrate by parts

1, A

2, C

1, D

$$\lim_{M \rightarrow \infty} \int_{x_0}^M \frac{\log x}{x^2} dx = \lim_{M \rightarrow \infty} \left\{ \left[-\frac{\log x}{x} \right]_{x_0}^M + \int_{x_0}^M \frac{1}{x^2} dx \right\} < \infty.$$

Hence there are also no issues as $x \rightarrow \infty$.

Hence the improper integral I exists.

2. (a) (i) Let $\sqrt{\xi} = y$, therefore $(1/2)d\xi/\sqrt{\xi} = dy$ and the integral becomes

sim. seen ↓

- (ii) Rate of change of volume equals the rate at which it leaves through the hole, plus the rate at which it leaves through the wetted curved wall. The hole loses water at a rate equal to $\pi r^2 \sqrt{gh}$ and the wall at a rate $k2\pi Rh$. Hence since the volume decreases $dV/dt < 0$ and the given equation follows. The dimensions of dV/dt are L^3/T and of the loss term kL^2 , hence k has dimensions of velocity, i.e. L/T .
- (iii) At a given t we have $V = \pi R^2 h(t)$, hence the equation becomes, after dividing by πR^2

2, A

sim. seen ↓

1, A

1, B

1, C

sim. seen ↓

1, A

1, B

2, D

$$\frac{dh}{dt} = -\frac{r^2 \sqrt{g}}{R^2} \left[\sqrt{h} + \frac{2kR}{r^2 \sqrt{g}} h \right], \quad h(0) = h_0.$$

Separate variables and integrate

$$\int_{h_0}^h \frac{dh}{\sqrt{h} + \frac{2kR}{r^2 \sqrt{g}} h} = -\frac{r^2 \sqrt{g}}{R^2} t,$$

and the integral on the left is of the form in part (a)(i) above with $A := \alpha = \frac{2kR}{r^2 \sqrt{g}}$. Hence

$$\frac{2}{\alpha} \log \left(\frac{1 + \alpha \sqrt{h}}{1 + \alpha \sqrt{h_0}} \right) = -\frac{r^2 \sqrt{g}}{R^2} t.$$

Re-arranging gives $\frac{1 + \alpha \sqrt{h}}{1 + \alpha \sqrt{h_0}} = \exp(-kt/R)$, i.e.

$$\sqrt{h} = \frac{(1 + \alpha \sqrt{h_0})e^{-kt/R} - 1}{\alpha}, \quad \alpha = \frac{2kR}{r^2 \sqrt{g}}.$$

The tank empties at $t = t_e$ given by $h(t_e) = 0$, i.e. $t_e = \frac{R}{k} \log(1 + \alpha \sqrt{h_0})$ from the solution above.

unseen ↓

- (iv) As $k \rightarrow 0$ we also have $\alpha \rightarrow 0$ and the solution for $\sqrt{h}$ is of the form $0/0$. Use L'Hôpital's rule noting that $\alpha = \frac{2kR}{r^2 \sqrt{g}}$ to find the limiting form

2, C

1, D

$$\sqrt{h} = \lim_{k \rightarrow 0} \frac{\left(1 + \frac{2kR\sqrt{h_0}}{r^2 \sqrt{g}}\right) (-t/R)e^{-kt/R} + e^{-kt/R} \frac{2R\sqrt{h_0}}{r^2 \sqrt{g}}}{\frac{2R}{r^2 \sqrt{g}}} = \sqrt{h_0} \left(1 - \frac{r^2 \sqrt{g}}{2R^2 \sqrt{h_0}} t\right)$$

Hence $t_s = \frac{2R^2 \sqrt{h_0}}{r^2 \sqrt{g}}$. Since more water is lost when $k \neq 0$, we would expect $t_e < t_s$.

seen ↓

- (b) (i) The function is neither even nor odd, hence

2, A

2, B

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

where

$$a_n = \frac{1}{\pi} \int_0^{\pi} \cos(nx) dx, \quad b_n = \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx,$$

i.e.

$$a_0 = 1, \quad a_n = 0, \quad b_n = \frac{1}{n\pi} (1 - \cos n\pi),$$

$$g(x) = \frac{1}{2} + \frac{2}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right)$$

- (ii) At points of discontinuity the Fourier series converges to the average, i.e. $g(x) = \frac{1}{2}(g(x^+) + g(x^-))$. Hence we expect $g(0) = 1/2$, $g(\pi) = 1/2$, $g(-121\pi) = 1/2$. Check these from the Fourier series for $g(x)$ by inserting $x = 0, \pi, -121\pi$, clearly all arguments of $\sin$ in the series are integer multiples of π and so $g(0) = g(\pi) = g(-121\pi) = 1/2$.

sim. seen ↓

1, B

1, D

- (iii) Pick $x = \pi/2$. We have $g(\pi/2) = 1$, hence

sim. seen ↓

2, A

$$1 = \frac{1}{2} + \frac{2}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right) \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Review of mark distribution:

Total A marks: 16 of 24 marks

Total B marks: 10 of 15 marks

Total C marks: 6 of 9 marks

Total D marks: 8 of 12 marks

Total Mastery marks: 0 of 20 marks

Total marks: 40 of 80 marks

3.(a(i)) We have for matrix A associated to the system of linear ODEs:

meth seen ↓

$$A = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}.$$

We calculate eigenvalues and eigenvectors of the matrix A :

$$\lambda_{1,2} = 1 \pm 2i \Rightarrow \vec{v}_{1,2} = \begin{pmatrix} 1 \\ \pm 2i \end{pmatrix},$$

These eigenvalues are complex. From the lecture we can write the general solution in terms of the eigenvalues and eigenvectors, so we have for the general solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{(1+2i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} + c_2 e^{(1-2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}.$$

Where, c_1 and c_2 are (complex) constants of integration. To obtain the solution in real form we write the solutions in terms of sin and cos functions.

3, B

$$\begin{aligned} x &= e^t [(c_1 + c_2) \cos 2t + (c_1 - c_2)i \sin 2t] \\ y &= e^t [2(c_1 - c_2)i \cos 2t - 2(c_1 + c_2) \sin 2t] \end{aligned}$$

We define new real constants of integration $A = c_1 + c_2$ and $B = (c_1 - c_2)i$ and we obtain:

$$\begin{aligned} x_{GS} &= e^t [A \cos 2t + B \sin 2t] \\ y_{GS} &= e^t [2B \cos 2t - 2A \sin 2t] \end{aligned}$$

(a(ii)) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an unstable fixed point as the real part of the eigenvalues are positive. The vector field on the x axis can be calculated by:

2, B

meth seen ↓

$$A \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ -4x \end{pmatrix}.$$

Figure 1 shows the phase portrait for this system, which is of the form of an unstable spiral and the direction of the rotation is determined using the vector field evaluated above.

4, A

(a(iii)) We try the following ansatz $\begin{pmatrix} a + be^t \\ c + de^t \end{pmatrix}$ using the method of undetermined coefficients. Plugging this into the ODE we have:

meth seen ↓

$$\frac{d}{dt} \begin{pmatrix} a + be^t \\ c + de^t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} a + be^t \\ c + de^t \end{pmatrix} + \begin{pmatrix} e^t \\ 5 \end{pmatrix}.$$

We obtain $a = 1$, $b = 0$, $c = -1$ and $d = -1$. So we obtain the following general solutions:

$$\begin{aligned} x_{GS} &= e^t [A \cos 2t + B \sin 2t] + 1 \\ y_{GS} &= e^t [2B \cos 2t - 2A \sin 2t] - 1 - e^t \end{aligned}$$

4, D

(b(i)) We will write ODE in the following form

seen ↓

$$F(x, y)dx + G(x, y)dy = 0.$$

We have $F = 3y \log y + 3y^2$ and $G = x + xy$. Then we test if the ODE is exact using the condition of integrability:

$$\frac{\partial F}{\partial y} = 3 \log y + 3 + 6y \neq \frac{\partial G}{\partial x} = 1 + y.$$

So the ODE is not exact

2, A

(b(ii)) As suggested we consider an integrating factor of the form $\lambda = x^m y^n$. We then try to satisfy the condition of integrability.

unseen ↓

$$\frac{\partial(\lambda F)}{\partial y} = 3(n+1)x^m y^n \log y + 3x^m y^n + 3(2+n)y^{1+n}x^m.$$

$$\frac{\partial(\lambda G)}{\partial x} = (m+1)x^m y^n + (m+1)x^m y^{1+n}.$$

By equating these two expressions, we find firstly that $n = -1$ as needed to set the coefficient of the term with $\log y$ to zero and then we find $m = 2$ will result in other terms to be equal in the two expressions. Therefore $\lambda = x^2/y$ is an integrating factor. Using this result, assuming implicit solutions $u(x, y) = 0$, we obtain:

3, C

$$\frac{\partial u}{\partial x} = \lambda F \quad \rightarrow \quad u = x^3(\log y + y) + h(y)$$

. Then we have

$$\frac{\partial u}{\partial y} = \lambda G \quad \Rightarrow \quad u = x^3(\log y + y) + c.$$

So, the solution of this ODE in implicit form is:

$$u(x, y) = x^3(\log y + y) + c = 0.$$

2, A

4. (a) The fixed points of the system are obtained by putting the right hand side to zero. We have

meth seen ↓

$$f(y) = ry - \frac{y}{1+y} = 0 \Rightarrow y_1^* = 0, \quad y_2^* = \frac{1}{r} - 1$$

So, we have two fixed points. We also note that we have singularity at $y_{sing} = -1$ as the right hand side is not defined for this value of y .

2, A

We consider three cases of $r < 0$, $r > 1$ and $0 < r < 1$. In Figure 2, we have determined the stability of these fixed points and singular point for the different values of r by plotting the two parts of $f(y)$, ry and $\frac{y}{1+y}$ against y . (note that one could also plot the limiting cases of $r = 0$ and $r = 1$ as a separate case as in this case there are only one fixed point at 0). The figure shows that $y_{sing} = -1$ is always unstable but the stability of fixed points depends on the value of r as seen in Figure 2.

3, C

- (a(ii)) The bifurcation diagram for this system is sketched in Figure 3. There is a transcritical bifurcation at $r = 1$. For $r = \frac{1}{4}$, the $y = 0$ is a stable fixed point and as seen in Figure 3 solutions approach this fixed point if initial condition is between -1 and 3 .

meth seen ↓

2, D

- (b) Taking Fourier transform from both sides of the ODE and using the linearity and properties of Fourier transforms of the derivative from the lecture we will obtain the following ODE for the Fourier transform of $y(x)$ denoted as $\hat{y}(\omega)$:

1, D

1, D

seen ↓

$$-\omega^2 \hat{y}(\omega) - i \frac{d\hat{y}(\omega)}{d\omega} = 0.$$

This is a first order separable ODE and by solving it we obtain:

$$\hat{y}(\omega) = ce^{\frac{i\omega^3}{3}},$$

where c is an arbitrary constant of integration. Using the inverse transform we obtain:

$$y(x) = \frac{c}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega x + \omega^3/3)} d\omega = \frac{c}{\pi} \int_0^{\infty} \cos(\omega x + \omega^3/3) d\omega.$$

5, B

- (c) From the lectures the Taylor expansion of $f(x, y)$ around the point $\vec{x}_0 = (x_0, y_0)$ in terms of Gradient, $\vec{\nabla} f(\vec{x}_0)$ and Hessian, $H_{ij}(\vec{x}_0)$ we have:

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + \vec{\nabla} f(\vec{x}_0) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \Delta x & \Delta y \end{pmatrix} H \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + \dots$$

We consider $z(x, y)$ given in implicit form:

2, A

$$F(x, y, z) = \sin(xz) - z - yx = 0.$$

We note firstly that at $x = 0$ and $y = 1$, we have $z = 0$. Then as z is available in implicit form we need to use implicit differentiation to evaluate $\vec{\nabla} z(0, 1)$. We have

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{-z \cos(xz) + y}{x \cos(xz) - 1}.$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \frac{x}{x \cos(xz) - 1}.$$

Evaluating these at $(0, 1)$, we obtain up to the first order:

$$z(\Delta x, 1 + \Delta y) = z(0, 1) + \frac{\partial z}{\partial x}(0, 1)\Delta x + \frac{\partial z}{\partial y}(0, 1)\Delta y + \cdots = -\Delta x + \dots$$

4, A

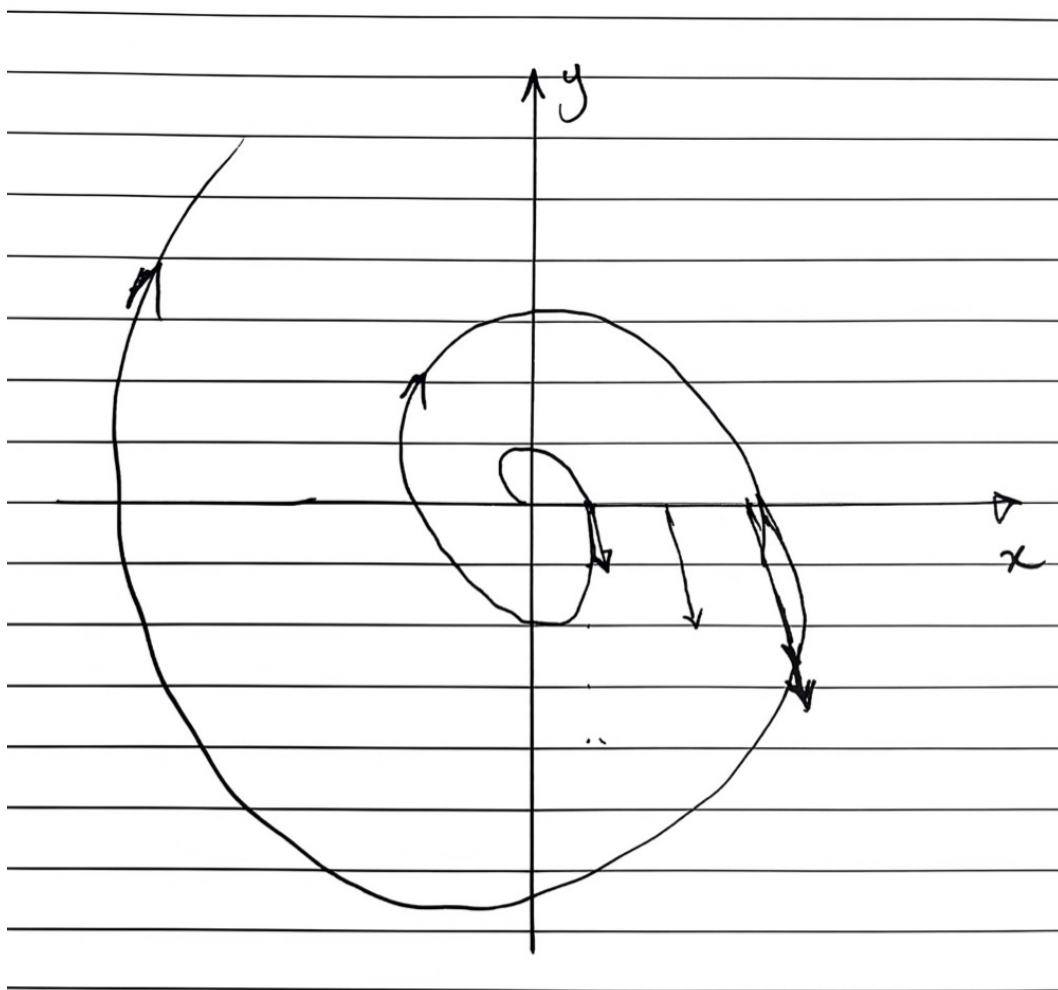


Figure 1: Phase portrait related to Problem 3a

Review of mark distribution:

Total A marks: 16 of 24 marks

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Total Mastery marks: 0 of 20 marks

Total marks: 40 of 80 marks

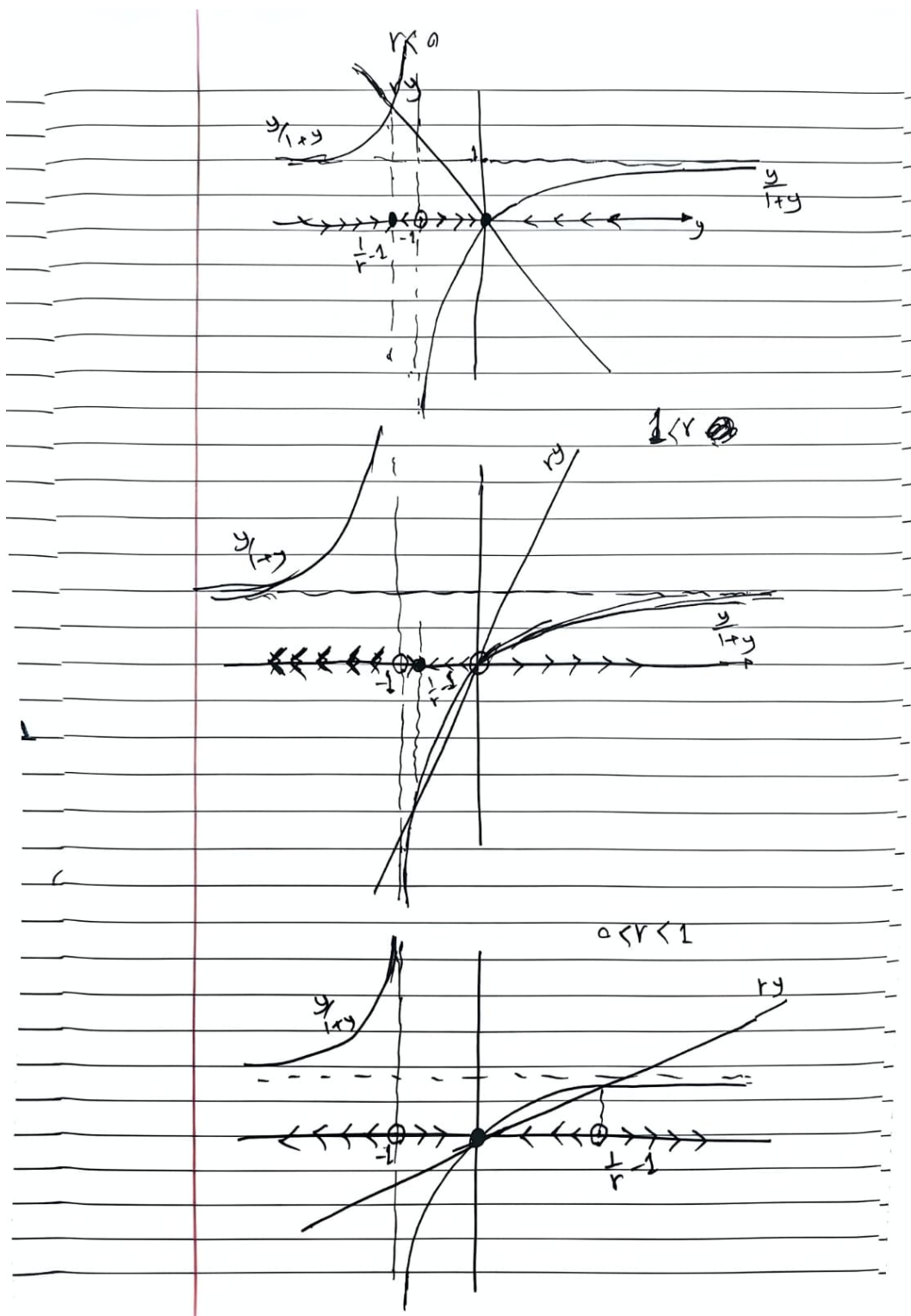


Figure 2: Figures for stability of fixed points related to Problem 4a(i)

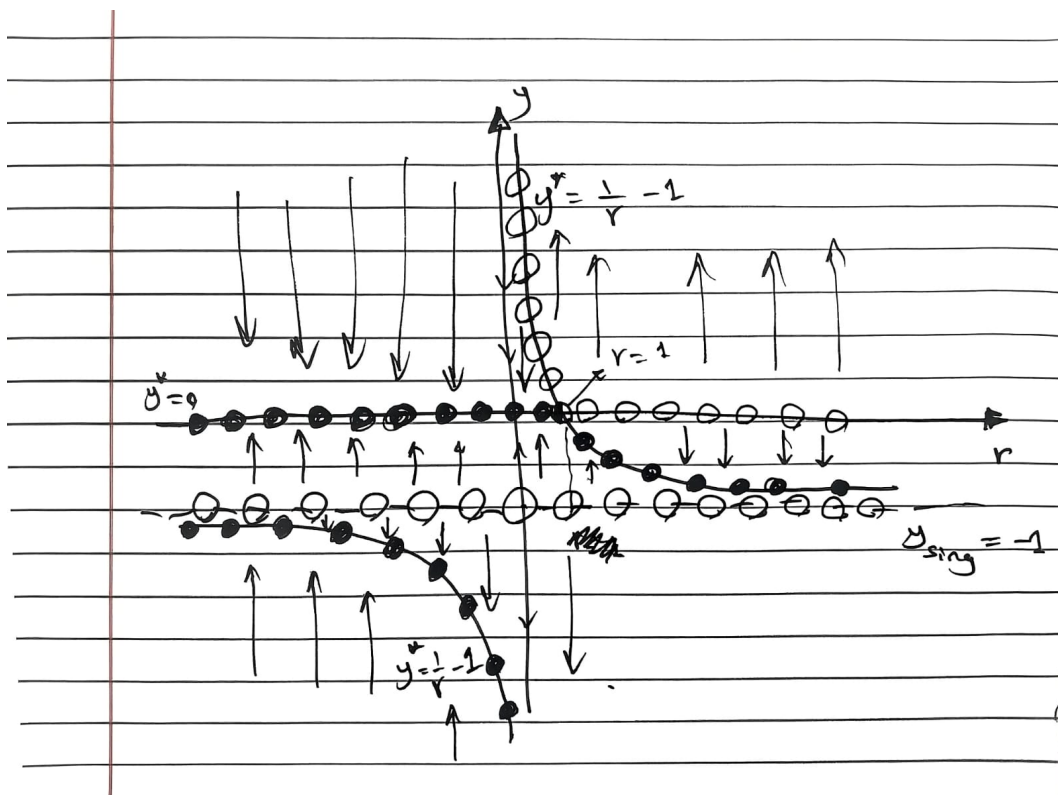


Figure 3: Figure for bifurcation diagram related to Problem 4a(ii)