

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Calculus and Applications

Date: Tuesday, 11 May 2021

Time: 09:00 to 12:00

Time Allowed: 3 hours

Upload Time Allowed: 45 minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. The function $f(x)$ is defined as follows: It is the piecewise linear function $y = f(x)$ that takes the values $f(\pm 1/2^n) = (-1/2)^n$ for all positive integers n and is linear for intermediate values of x .
- (a) Determine how to define $f(0)$ in order to make $f(x)$ continuous at $x = 0$. (3 marks)
 - (b) Sketch $f(x)$ in the interval $-1/2 \leq x \leq 1/2$ and discuss what happens as $x \rightarrow 0$. (5 marks)
 - (c) Sketch $f'(x)$ in the interval $-1/2 \leq x \leq 1/2$ and discuss what happens as $x \rightarrow 0$. Does $f'(0)$ exist? (6 marks)
 - (d) Find the length of the curve sketched in part (c) above. (6 marks)

(Total: 20 marks)

2. In this question the following physical rule will be useful: *The weight of a solid object floating in a fluid of constant density is equal to the weight of fluid it displaces.* [Note: Mass is equal to volume times density, and the weight is the mass times the acceleration of gravity].
- (a) The graph of the function $y = x^2/a$, for $0 \leq x \leq a$ where $a > 0$ is a constant, is rotated about the y -axis to generate a bowl $\mathcal{B}$ of height a that is open at the top.
 - (i) Find the volume of the bowl. (4 marks)
 - (ii) Find the surface area of the bowl. (4 marks)
 - (b) A bowl such as $\mathcal{B}$ is made out of steel that has density ρ_s per unit area (the units of ρ_s are kg/m^2). When carefully placed vertex down in a bath of water of density $\rho_w \text{ kg/m}^3$, it floats with its axis vertical and partly submerged under water.
 - (i) If half its height is submerged, find ρ_s in terms of ρ_w . (3 marks)
 - (ii) What is the largest ρ_s that can be used to ensure the bowl does not sink. (3 marks)
 - (c) The bowl $\mathcal{B}$ is now made out of very light plastic (i.e. assume $\rho_s = 0$) and filled completely with a fluid that has sediments added to it so that its density ρ varies with the height from the bowl's bottom. Experiments suggest that $\rho(y) = \rho_w(2 - \beta y/a)$, where $\beta > 0$ is a constant, is a good model. Find physically admissible upper and lower bounds for β that ensure that the bowl does not sink. (6 marks)

(Total: 20 marks)

3. (a) Given a positive integer n and a constant a , we define the integral

$$I_n = \int_0^1 x^a (\log x)^n dx.$$

- (i) Determine for what values of a the integral exists. (2 marks)
(ii) If a is such that I_n exists, find a recursion relation for I_n and use it to calculate I_5 . (4 marks)

- (b) Show that $\frac{d}{dx} \tan x = \sec^2 x$. Hence, or otherwise, calculate

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \sec^2 \frac{\pi}{4n} + \sec^2 \frac{2\pi}{4n} + \dots + \sec^2 \frac{n\pi}{4n} \right).$$

(6 marks)

- (c) (i) Consider the function $f(x) = \sin(x^2/\pi)$. Find $f'(x)$ and sketch $f(x)$ in the interval $-\pi \leq x \leq \pi$. (3 marks)
(ii) Extend $f(x)$, $-\pi \leq x \leq \pi$, periodically to obtain the 2π -periodic function $g(x)$. Find the Fourier series, $F(x)$ say, of $g(x)$ [give formulas for the Fourier coefficients but do not attempt to compute them]. (3 marks)
(iii) What values do $F(x)$ and $F'(x)$ converge to at $x = 0$ and $x = \pi$? Justify your answer. (2 marks)

(Total: 20 marks)

4. (a) Consider the following linear ODE:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = f(x),$$

where,

$$f(x) = \begin{cases} x^2 e^{-3x}, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

- (i) Find the general solution to the corresponding homogenous ODE using an appropriate basis for the solution space and prove the linear independence of the basis function chosen. (4 marks)
- (ii) Find the solution to the initial value problem for the full non-homogenous ODE for all x , given $y(-1) = \frac{dy}{dx}(-1) = 0$. Show the steps of your derivation. (7 marks)
- (b) Consider the function $f(x)$ defined over $(-\infty, \infty)$. We are given the Taylor expansion of $\hat{f}(\omega)$ (Fourier transform of $f(x)$) near $\omega = 0$ up to the third order:

$$\hat{f}(\omega) = i\omega + \frac{i}{3}\omega^3 + \dots$$

Obtain the value of the following integral with this information.

$$\int_{-\infty}^{\infty} x^3 f(x) dx.$$

(4 marks)

- (c) Use the energy theorem for Fourier transforms to evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{d\omega}{a^2 + \omega^2}.$$

(5 marks)

(Total: 20 marks)

5. (a) Consider the following System of linear ODEs:

$$\begin{aligned}\frac{dx}{dt} &= x - 2y \\ \frac{dy}{dt} &= 3x - 6y\end{aligned}$$

- (i) Find the general solution of this system of ODEs, obtain its fixed point(s) and describe their stability. What is the asymptotic behavior when $x(0) = -1$ and $y(0) = 2$? (6 marks)
- (ii) Obtain the general solution of this modified system:

$$\begin{aligned}\frac{dx}{dt} &= x - 2y + 6e^t, \\ \frac{dy}{dt} &= 3x - 6y.\end{aligned}$$

(3 marks)

- (b) Consider the phase portrait of a nonlinear system of ODEs for $(x(t), y(t))$ shown in Figure 1 (appearing in the next page) that illustrates the vector field and some trajectories. Identify the coordinates of the fixed point(s) and describe their stability. What type of phase portrait behaviour does the linearised system exhibit near the fixed point(s)? (6 marks)
- (c) Consider the following ODE:

$$\frac{dx}{dt} = \alpha + x^4 - 1.$$

Obtain the fixed points of the system, discuss their stability and sketch the bifurcation diagram of this ODE. Identify any bifurcation point(s) and state the bifurcation type(s). (5 marks)

(Total: 20 marks)

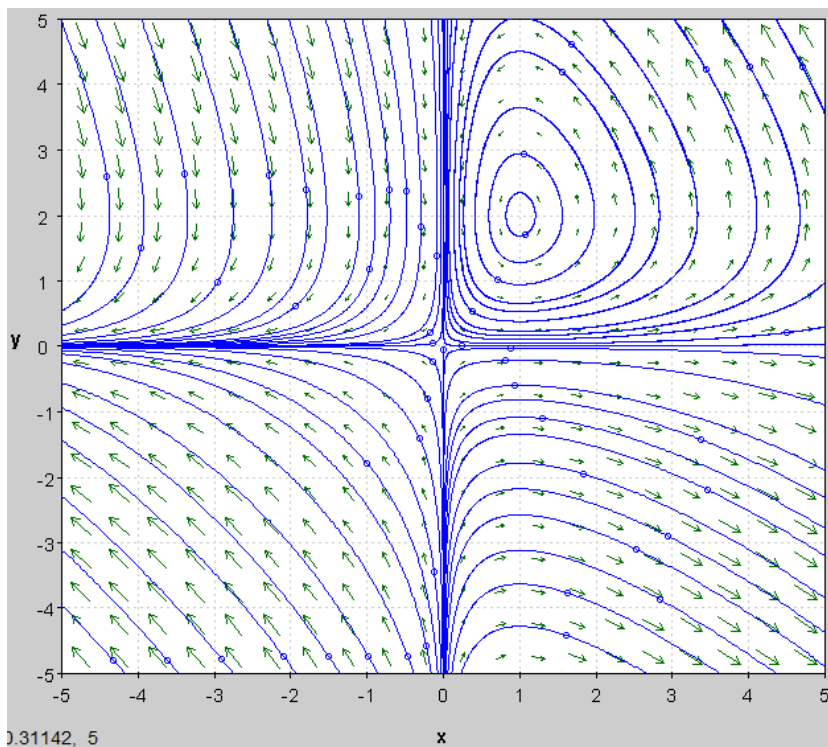


Figure 1:

6. (a) Consider the following ODE:

$$\left(\frac{y^2}{x} + 3y^3\right)dx + \left(\frac{1}{x} - y\right)dy = 0.$$

Show that this ODE is not exact. Then find an integrating factor of the form

$$\lambda(x, y) = x^m y^n,$$

where m, n are constants in $\mathbb{R}$ to obtain a solution of this ODE in implicit form.

(8 marks)

- (b) Consider the function

$$u(x, y) = x^3y - xy^2 - 4xy.$$

- (i) Find the set of points (x_c, y_c) for which $u(x_c, y_c) = 0$ (i.e., the level curve of $u = 0$).
(2 marks)
- (ii) Find all the extrema (stationary points) of u and use the Hessian to decide the character of the stationary points.
(6 marks)
- (iii) Sketch u in the x, y plane, highlighting some of the level curves, stationary points obtained above and also the sign of the function u . For the point $(0, 0)$, reason why what you found in the last part is consistent with your sketch.
(4 marks)

(Total: 20 marks)

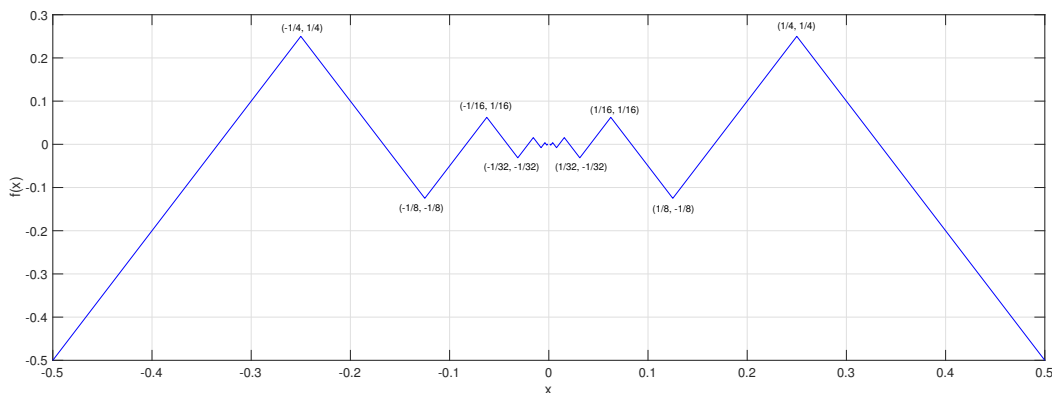


Figure 1: Figure to problem 1(b)

1. (a) As $n \rightarrow \infty$ we have $f(\pm 1/2^n) \rightarrow 0$. Hence need $f(0) = 0$ to make the function continuous. sim. seen ↓
- (b) It is enough to identify the points $(\pm 1/2^n, (-1/2)^n)$ on the graph for $n \geq 1$, and join straight lines between successive ones. The function is symmetric, so enough to plot $x > 0$. Here are the first few: $(1/2, -1/2), (1/4, 1/4), (1/8, -1/8), \dots$. The plot is given in Figure 1. As $x \rightarrow 0$ the function tends to zero and the piecewise linear intervals decrease. 2, A
sim. seen ↓
- (c) The function $f'(x)$ is discontinuous. The function is now odd, so looking at $x > 0$. The slope in the interval I_n given by $\frac{1}{2^{n+1}} < x < \frac{1}{2^n}$ is 5, B
1, C
unseen ↓

$$f'_{I_n} = \frac{\left(\frac{(-1)^n}{2^n} - \frac{(-1)^{n+1}}{2^{n+1}}\right)}{\left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)} = \frac{\frac{(-1)^n}{2^{n+1}}(2+1)}{\frac{1}{2^{n+1}}(2-1)} = (-1)^n 3, \quad \frac{1}{2^{n+1}} < x < \frac{1}{2^n}.$$

The sketch is given in Figure 2. 4, C

As $n \rightarrow \infty$ and $x \rightarrow 0$, the derivative is either 3 or -3 hence cannot exist. 2, C

- (d) The function value of $f'(x)$ is equal to ± 3 (i.e. a constant) and this holds for the union of all intervals $\frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}$ and $-\frac{1}{2^n} \leq x \leq -\frac{1}{2^{n+1}}$ for $n \geq 1$. Hence the length of the curve is simply the size of the interval $[-\frac{1}{2}, \frac{1}{2}]$, i.e. 1. sim. seen ↓

[Note: Students may not see this and form the sum $2 \sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right)$ which telescopes to give $2\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} + \dots\right) = 1$.] 6, B

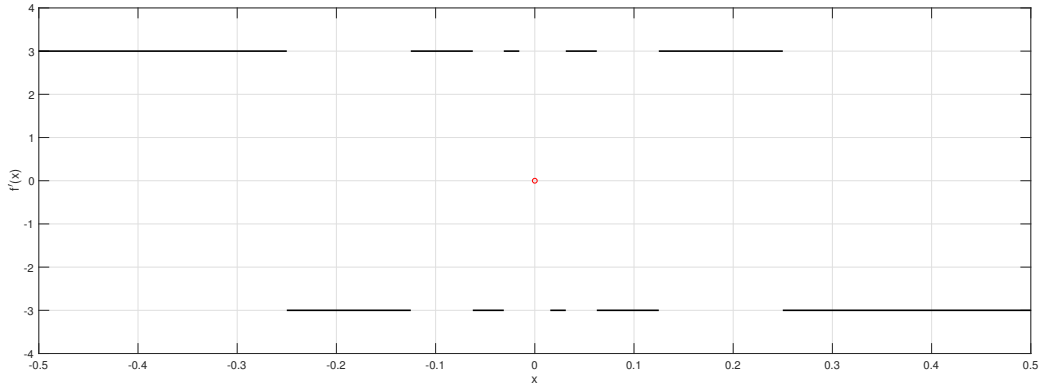


Figure 2: Figure to problem 1(c)

2. (a) (i)

$$V = \int_0^a \pi x^2 dy = \int_0^a \pi a y dy = \frac{1}{2} \pi a^3$$

seen ↓

(ii)

$$\begin{aligned} S &= \int_0^a 2\pi x ds = \int_0^a 2\pi x \sqrt{dx^2 + dy^2} = 2\pi \int_0^a x \sqrt{(dx/dy)^2 + 1} dy \\ &= 2\pi \int_0^a \sqrt{ay} \sqrt{\frac{a}{4y} + 1} dy = a^{1/2} \pi \int_0^a \sqrt{a + 4y} dy = \frac{1}{6} a^{1/2} \pi \left[(a + 4y)^{3/2} \right]_0^a \\ &= \frac{1}{6} (5^{3/2} - 1) \pi a^2. \end{aligned}$$

4, A

seen ↓

4, A

Note of alternative solution: After the second step, the integration could be done with respect to x , i.e.

$$\begin{aligned} S &= 2\pi \int_0^a x \sqrt{(dy/dx)^2 + 1} dx = \frac{2\pi}{a} \int_0^a x \sqrt{4x^2 + a^2} dx \\ &= \frac{\pi}{6a} \left[(4x^2 + a^2)^{3/2} \right]_0^a = \frac{\pi}{6} a^2 (5^{3/2} - 1). \end{aligned}$$

unseen ↓

- (b) (i) We are given that $a/2$ is the submerged height. The volume of water displaced by the floating bowl is equal to the submerged volume. Hence the weight of the bowl, using the given physical rule, must be

3, D

$$W = \rho_w g \int_0^{a/2} \pi a y dy = \frac{\pi}{8} \rho_w g a^3.$$

But we are also given the density of the bowl's surface area, hence using the result from 2(a)(ii) we have

$$\frac{\pi}{8} \rho_w g a^3 = \rho_s g \frac{\pi}{6} a^2 (5^{3/2} - 1) \quad \Rightarrow \quad \rho_s = \frac{3\rho_w a}{4(5^{3/2} - 1)}.$$

unseen ↓

- (ii) The critical density ρ_s must be such that the whole volume $\pi a^3/2$ is submerged, hence

3, D

$$\frac{1}{2} \pi a^3 \rho_w = \rho_s \frac{\pi}{6} a^2 (5^{3/2} - 1) \quad \Rightarrow \quad \rho_s = \frac{3\rho_w a}{(5^{3/2} - 1)}.$$

unseen ↓

- (c) Suppose the bowl is floating and is only just not sinking. This means that the weight of water the bowl is displacing must equal the weight of sediment fluid it is

6, D

carrying when completely full. The equation is

$$\begin{aligned}\frac{1}{2}\pi\rho_w a^3 &= \int_0^a \pi a y \rho_w \left(2 - \frac{\beta y}{a}\right) dy \Rightarrow \\ \frac{1}{2}\pi\rho_w a^3 &= \pi a^3 \rho_w \left(1 - \frac{\beta}{3}\right) \Rightarrow \beta = \frac{3}{2}\end{aligned}$$

Hence if $\beta < 3/2$ the bowl will sink (e.g. $\beta = 0$), hence we require $\beta > 3/2$ so that the density reduces sufficiently fast to control the weight. To find an upper bound of admissible β we observe that the density $\rho(y = a) = \rho_w(2 - \beta) > 0$, hence $\beta < 2$. The final answer then is

$$\frac{3}{2} \leq \beta \leq 2.$$

[Note: Give 4 marks for the lower bound and 2 marks for the physical upper bound.]

3. (a) (i) The only problem is as $x \rightarrow 0+$. Clearly if $a \leq 0$ the integrand diverges. Hence we must have $a > 0$ but need to show boundedness. Use L'Hopital's rule in the form

seen ↓

2, A

$$\lim_{x \rightarrow 0+} \frac{(\log x)^n}{x^{-a}} = \lim_{x \rightarrow 0+} \frac{n(\log x)^{n-1}}{x(-a)x^{-a-1}} = \dots = \lim_{x \rightarrow 0+} \frac{n!}{(-a)^n x^{-a}} = 0.$$

meth seen ↓

- (ii) Integrate by parts

4, A

$$I_n = \left[\frac{x^{a+1}}{(a+1)} (\log x)^n \right]_0^1 - \int_0^1 \frac{x^{a+1}}{(a+1)x} n(\log x)^{n-1} dx = -\frac{n}{(a+1)} I_{n-1}$$

Hence

$$I_5 = -\frac{5}{(a+1)} I_4 = \frac{5 \cdot 4}{(a+1)^2} I_3 = -\frac{5 \cdot 4 \cdot 3}{(a+1)^3} I_2 = \frac{5 \cdot 4 \cdot 3 \cdot 2}{(a+1)^4} I_1 = -\frac{5!}{(a+1)^5} \int_0^1 x^a dx \\ = -\frac{5!}{(a+1)^6}$$

- (b) Use the quotient rule

seen ↓

2, A

$$\frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \sec^2 x$$

This means that $\int \sec^2 x dx = \tan x$, and we see that the sum needed is the Riemann sum of the integral $\int_0^{\pi/4} \sec^2 x dx$, since splitting the interval into n points, i.e. the grid spacing is $h = \pi/(4n)$ we have

sim. seen ↓

4, B

$$\int_0^{\pi/4} \sec^2 x dx = \lim_{h \rightarrow 0} h (\sec^2 0 + \sec^2 h + \sec^2 2h + \dots + \sec^2 nh) \\ = \lim_{n \rightarrow \infty} \frac{\pi}{4n} \left(1 + \sec^2 \frac{\pi}{4n} + \sec^2 \frac{2\pi}{4n} + \dots + \sec^2 \frac{n\pi}{4n} \right) \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \left(1 + \sec^2 \frac{\pi}{4n} + \sec^2 \frac{2\pi}{4n} + \dots + \sec^2 \frac{n\pi}{4n} \right) = \frac{4}{\pi} [\tan x]_0^{\pi/4} = \frac{4}{\pi}$$

seen ↓

- (c) (i)

3, A

$$f'(x) = \frac{2x}{\pi} \cos(x^2/\pi).$$

The function $f(x)$ is even and $f'(0) = 0$. Also, $f(x) \geq 0$ in the given interval and $f'(\pi) = -2 = -f'(-\pi)$. A sketch is given in figure 3.

seen ↓

- (ii) As seen above the function $f(x)$ is even, hence the Fourier series of $g(x)$ must be a cosine series. Hence

3, A

$$F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_n = \frac{2}{\pi} \int_0^{\pi} \sin(x^2/\pi) \cos(nx) dx.$$

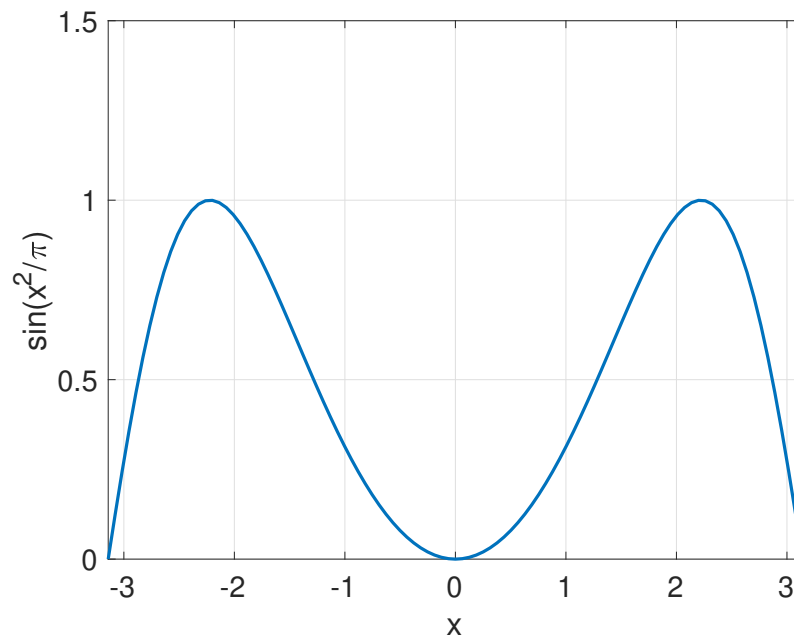


Figure 3: Figure to problem 3(c)(i)

- (iii) The function $g(x)$ is continuous, hence $F(0) = f(0) = 0$ and $F(\pi) = f(\pi) = 0$.

Since $g(x)$ is continuous then the series $F(x)$ is uniformly convergent. Hence, $F'(x)$ will converge to $g'(x)$ uniformly *except* at points where $g'(x)$ is discontinuous, and at such points it converges to the average value of the discontinuity. Hence $F'(0) = 0$ since $g'(0) = 0$ and is continuous there. But, $g'(\pi-) = -2$ and $g'(\pi+) = g'(-\pi+) = 2$ (by periodicity). Hence $F'(\pi) = 0$.

sim. seen ↓

2, C

4. (a-i) We have, repeated root $\lambda = -3$ for the charecteristic equation for the correspondong homogenous ODE, so:

seen ↓

$$y_{CF} = c_1 e^{-3x} + c_2 x e^{-3x}.$$

2, A

We show the Wronskian is non-zero to prove these two solutions are linearly independent:

$$W(x) = \det \begin{bmatrix} e^{\lambda x} & x e^{\lambda x} \\ \lambda e^{\lambda x} & e^{\lambda x} + \lambda x e^{\lambda x} \end{bmatrix} = e^{2\lambda x} \neq 0.$$

2, A

- (a-ii) We solve the ODE in two steps. First, for $x < 0$, we use the following Ansatz (using the method of undetermined coefficents, alternatively, one can also use the method of variation of parameters):

sim. seen ↓

$$y_{PI} = A x^4 e^{-3x}.$$

Plugging this into the ODE we obtain $A = \frac{1}{12}$.

2, A

Now, using the initial conditions of $y(-1) = y'(-1) = 0$, we obtain c_1 and c_2 , for $x < 0$:

$$y(x) = \frac{1}{12} e^{-3x} (x^4 + 4x + 3).$$

2, A

Next, we obtain the particular integral for $x > 0$ case, which is simply $y_{PI} = \frac{1}{9}$. Now using the values of $y(0) = \frac{1}{4}$ and $y'(0) = -\frac{5}{12}$, from the solution for $x < 0$ as the inital condition for the $x > 0$ we obtain new values for c_1 and c_2 . The solution for $x > 0$ is:

$$y(x) = \frac{5}{36} e^{-3t} + \frac{1}{9}.$$

3, C

- (b) Given the defintion of Fourier transform and taking derivatives with respect to ω we have

unseen ↓

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx, \quad (1)$$

$$\frac{d^3 \hat{f}(\omega)}{d\omega^3} = \int_{-\infty}^{\infty} f(x) (-ix)^3 e^{-i\omega x} dx \quad (2)$$

By setting $\omega = 0$, we obtain

$$\int_{-\infty}^{\infty} x^3 f(x) dx = -i \frac{d^3 \hat{f}(0)}{d\omega^3}$$

As we are given the Taylor expansion of $\hat{f}(\omega)$ near 0, so we can obtain the third derivative with respect to ω at 0, which is $2i$. So the integral is equal to 2.

4, D

(c) The energy theorem states:

unseen ↓

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} [f(x)]^2 dx.$$

Using $\hat{f}(\omega) = \frac{1}{a+i\omega}$, we would have the desired integral on the LHS of the energy theorem. This is the fourier transform of the following function from the problem sheet 1:

$$f(x) = \begin{cases} e^{-ax}, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases}$$

So, we have:

$$\int_{-\infty}^{\infty} \frac{d\omega}{a^2 + \omega^2} = 2\pi \int_0^{\infty} e^{-2ax} dx = \frac{\pi}{a}.$$

5, B

5. (a-i) The eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = 0$ and corresponding eigenvectors are

meth seen ↓

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \vec{v}_1 e^{-5t} + c_2 \vec{v}_2.$$

3, A

Fixed points are the line of $c_2 \vec{v}_2$, which are stable. The initial conditions given are satisfied with $c_1 = 1$ and $c_2 = -1$. Therefore asymptotically the solution decays along $\vec{v}_1$ to $-\vec{v}_2$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

3, A

- (a-ii) We try the following ansatz:

unseen ↓

$$\begin{bmatrix} x_{PI} \\ y_{PI} \end{bmatrix} = \begin{bmatrix} ae^t \\ be^t \end{bmatrix}.$$

We plug this into the ODE and solve for a and b and obtain:

$$a = a - 2b + 6,$$

$$b = 3a - 6b.$$

and we obtain $a = 7$ and $b = 3$.

3, C

- (b) The fixed points are $(0, 0)$ and $(1, 2)$.

sim. seen ↓

2, A

$(0, 0)$ is unstable and the phase portrait near this point is of saddle-point type. $(1, 2)$ is a Lyapunov stable fixed point and the phase portrait near this point is of centre type.

4, D

- (c) The fixed points are:

sim. seen ↓

$$x^* = \pm(1 - \alpha)^{1/4}$$

So, there are no fixed points for $\alpha > 1$, one fixed point at $\alpha = 1$ and two for $\alpha < 1$. The stability of the fixed points can be obtained using the graph of dx/dt vs x for $\alpha < 1$ as seen in Figure 1. There is one saddle-node bifurcation at $\alpha = 1$. The bifurcation diagram can also be seen in Figure 1.

5, B

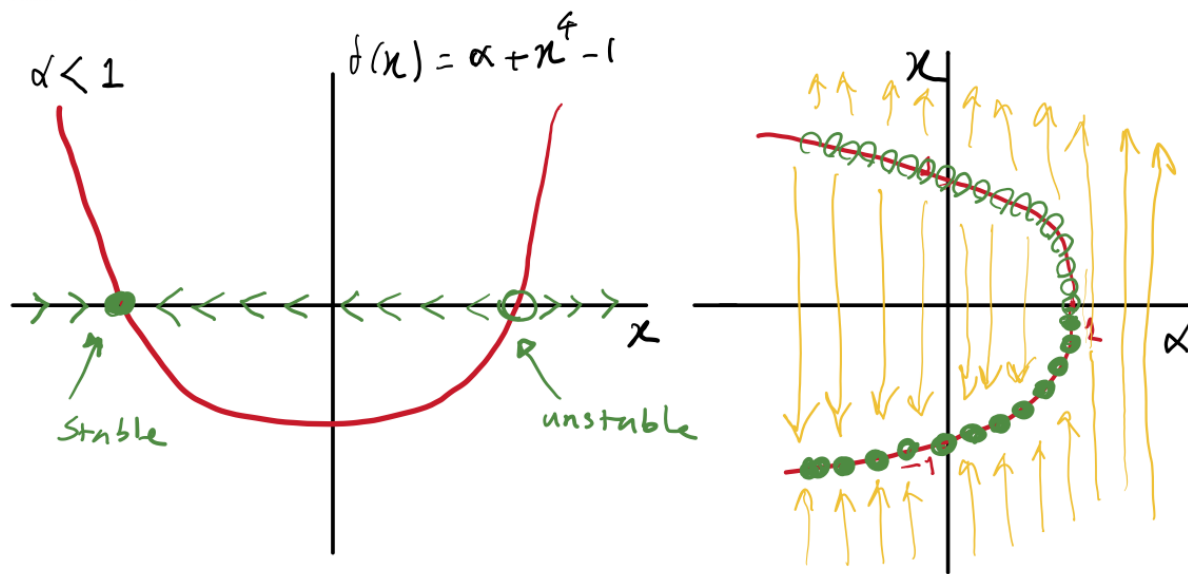


Figure 4:

6. (a) Let $F(x, y) = \frac{y^2}{x} + 3y^3$ and $G(x, y) = \frac{1}{x} - y$, so we obtain:

$$\frac{\partial F}{\partial y} = \frac{2y}{x} + 9y^2,$$

$$\frac{\partial G}{\partial x} = -\frac{1}{x^2}.$$

So, the condition of integrability is not satisfied.

Now, we multiply the ODE with $\lambda(x, y) = x^m y^n$. We have:

$$\frac{\partial F}{\partial y} = (n+2)x^{m-1}y^{n+1} + 3(n+3)x^m y^{n+2},$$

$$\frac{\partial G}{\partial x} = (m-1)x^{m-2}y^n - mx^{m-1}y^{n+1}.$$

By equating the term with the same powers on both sides, we get the following 3 equations:

$$-m = n + 2,$$

$$3(n+3) = 0,$$

$$m - 1 = 0.$$

$m = 1$ and $n = -3$ satisfies all the equations. So, we have $\lambda(x, y) = xy^{-3}$.

For the solution $u(x, y) = 0$, we have:

$$u(x, y) = \int G(x, y)xy^{-3} dy = -\frac{1}{2}y^{-2} + xy^{-1} + g(x).$$

So we have:

$$\frac{\partial u}{\partial x} = F(x, y)xy^{-3} \Rightarrow \frac{dg}{dx} = 3x \Rightarrow g(x) = \frac{3}{2}x^2 - c,$$

unseen ↓

2, B

2, B

2, D

where c is constant of integration. So we have:

$$u(x, y) = 0 \Rightarrow -\frac{1}{2}y^{-2} + xy^{-1} + \frac{3}{2}x^2 = c.$$

2, D

(b-i) We can write u as:

$$u = xy(x^2 - y - 4),$$

So, the level sets are $x_c = 0$, $y_c = 0$ and $y_c = x_c^2 - 4$.

2, A

(b-ii) We need to evaluate the gradient of u and set it to zero. We have:

sim. seen ↓

$$\frac{\partial u}{\partial x} = y(3x^2 - y - 4) = 0 \Rightarrow y = 0 \text{ or } 3x^2 - y - 4 = 0,$$

$$\frac{\partial u}{\partial y} = x(x^2 - 2y - 4) = 0 \Rightarrow x = 0 \text{ or } x^2 - 2y - 4 = 0.$$

Given the above, we have the following stationary points:

$$P_1 = (0, 0), \quad P_2 = (0, -4), \quad P_3 = (2, 0), \quad P_4 = (-2, 0), \quad P_5 = \left(\frac{2}{\sqrt{5}}, -\frac{8}{5}\right), \quad P_6 = \left(-\frac{2}{\sqrt{5}}, -\frac{8}{5}\right).$$

For the Hessian we have:

$$H(x, y) = \begin{bmatrix} 6xy & 3x^2 - 2y - 4 \\ 3x^2 - 2y - 4 & -2x \end{bmatrix}.$$

Now, we can evaluate Hessian at each stationary point and based on the value of trace and determinant of the matrix, we can decide if the stationary point is a maximum, minimum or saddle point.

$$H_{P_1} = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} \Rightarrow \tau = 0, \quad \Delta = -16 \Rightarrow P_1 \text{ is a saddle point.}$$

$$H_{P_2} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \Rightarrow \tau = 0, \quad \Delta = -16 \Rightarrow P_2 \text{ is a saddle point.}$$

$$H_{P_3} = \begin{bmatrix} 0 & 8 \\ 8 & -4 \end{bmatrix} \Rightarrow \tau = -4, \quad \Delta = -64 \Rightarrow P_3 \text{ is a saddle point.}$$

$$H_{P_4} = \begin{bmatrix} 0 & 8 \\ 8 & 4 \end{bmatrix} \Rightarrow \tau = 4, \quad \Delta = -64 \Rightarrow P_4 \text{ is a saddle point.}$$

$$H_{P_5} = \begin{bmatrix} -\frac{96}{5\sqrt{5}} & \frac{8}{5} \\ \frac{8}{5} & -\frac{4}{\sqrt{5}} \end{bmatrix} \Rightarrow \tau = -\frac{116}{\sqrt{55}}, \Delta = \frac{64}{5} \Rightarrow P_5 \text{ is a maximum.}$$

$$H_{P_6} = \begin{bmatrix} \frac{96}{5\sqrt{5}} & \frac{8}{5} \\ \frac{8}{5} & \frac{4}{\sqrt{5}} \end{bmatrix} \Rightarrow \tau = \frac{116}{\sqrt{55}}, \Delta = \frac{64}{5} \Rightarrow P_6 \text{ is a minimum.}$$

6, A

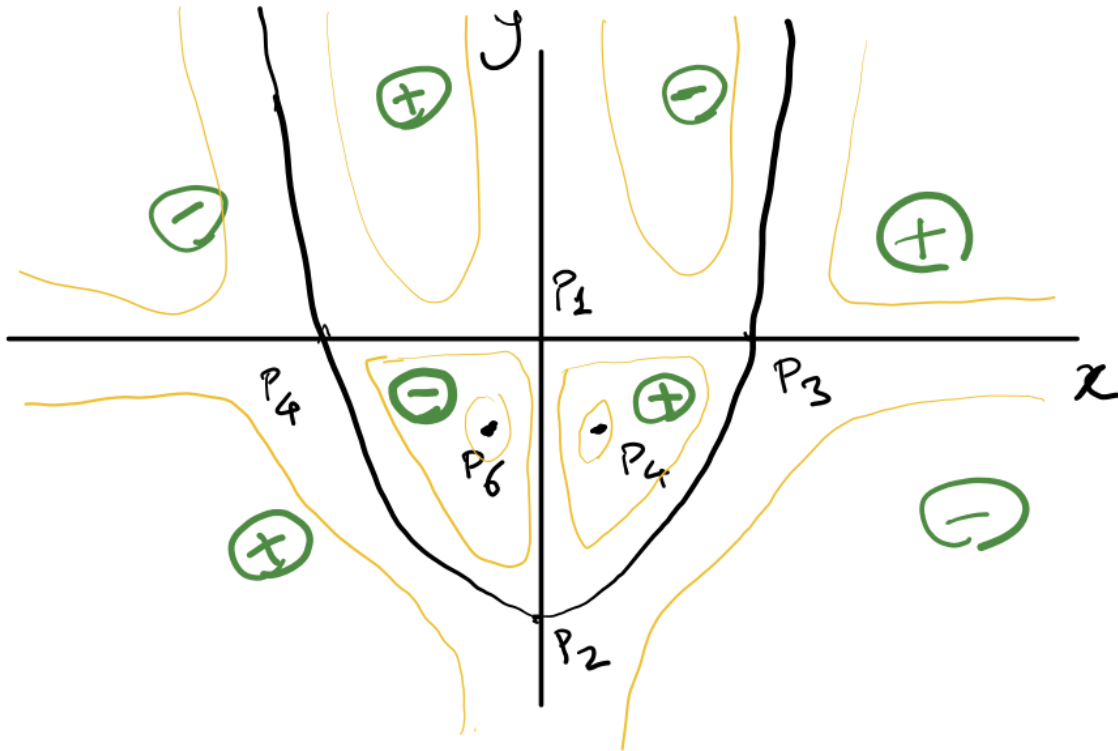


Figure 5:

- (b-iii) Figure 2 shows the sketch of the level sets of u , the stationary points and the sign of the function u .

3, C

For the point $P_1 = (0, 0)$, which based on the Hessian, we obtained to be a saddle point, we see that the value of the function u in the vicinity of the point P_1 in the $x = y$ direction becomes negative (decreases from 0) and in the $x = -y$ direction becomes positive (increases from 0), so this is consistent with point P_1 being a saddle-point.

1, B

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered.

For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add

ExamModuleCode	QuestionNumber	Comments for Students
MATH40004	1	Generally the question was done satisfactorily. I would encourage students to be more careful when sketching graphs - give pertinent coordinates and explain construction steps.
MATH40004	2	Most of students did well but didn't show steps and some used Maple so I deducted points. There was a common confusion regarding questions b and c, it seems that a lot of students are not good with word problems.
MATH40004	3	Most of students did well but didn't show steps and some used Maple so I deducted points.
MATH40004	4	There were 3 parts to this question. Only a handful of students answered all 3 parts correctly. Part a. was the most accessible/seen similar before and more or less complete answers were provided by most students. Parts b. and c. were less straightforward and required an extra step and/or better understanding of the material. A lot of students made good first attempts and received partial credit for it. A few students spotted an alternative direct/ quick way to compute the integral in Part c. by substitution.
MATH40004	5	Overall the question was very well answered, with scores of 17+ not being uncommon. Well done students! The full 20 marks was less regularly obtained due to some small details which all needed to be found. The first half of part (a) was very well done, with practically all students who looked for eigenvalues and eigenvectors of the matrix finding the correct values and the correct general solution. Some students tried to use alternative methods (solving a second order equation for x in terms of t and back substituting for y in terms of t), however these were often poorly implemented and correlated with students who performed weaker on the whole. The fixed points and their stability were regularly found correctly but few students went on to determine the asymptotic behaviour of the selected point in sufficient detail: it indeed tends to the line of stable fixed points, but to precisely the point (-2,-1) on this line. Part (a)(ii) was particularly disappointing, and probably the worst answered part of the question. Several students did indeed use the correct ansatz $(x,y) = (a,b)e^{at}$ and substitute this into the system correctly, however many students made errors here; the ansatz $(x,y) = (ae^{at}, 0)$ or $(x,y) = (ae^{at}, b)$ was a common error. Some students used the correct ansatz but failed to substitute into the equation correctly, often missing out the derivative term or the $+6e^{at}$ term for the first equation. This was also the part containing by far the most algebraic slips. Part (b) was regularly done well, with almost all students noticing the key points. Common ways to lose marks here were missing the fact that the saddle point is unstable or not mentioning the phase portrait types of the two points. Part (c) was also done well on the whole, with many correct or almost correct bifurcation diagrams being produced. A common slip here was not stating the values of the fixed points explicitly which was asked in the question, however many students correctly identified which branch was stable and which was unstable and correctly identified the bifurcation as a saddle-node bifurcation.
MATH40004	6	Most students have done well on this question particularly part a. Some students obtain $u(x,y)$ in part a, but don't write the implicit solution as $u = 0$. Also, in part b quite a few students don't obtain all the 6 stationary points.