

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Calculus and Applications

Date: 17 May 2022

Time: 09:00 – 12:00 (BST)

Time Allowed: 3:00 hours

Upload Time Allowed: 45 minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Consider the function $f(x) = 1 - e^{-1/x^2}$, $x \neq 0$.
- (i) Calculate $\lim_{x \rightarrow 0} f(x)$, $\lim_{x \rightarrow 0} f'(x)$, and $\lim_{x \rightarrow 0} f''(x)$. How should we define $f(0)$, $f'(0)$, $f''(0)$, so that f and its first two derivatives are continuous. (4 marks)
- (ii) By relying on your calculations above, present an argument that enables determination of $\lim_{x \rightarrow 0} f^{(n)}(x)$ where n is any positive integer. (2 marks)
- (b) (i) Find all inflection points of $f(x)$, $x \neq 0$, and its global maximum and minimum. (3 marks)
- (ii) Sketch $f(x)$ identifying the features you found above in part (b)(i). (3 marks)
- (c) Consider the integral $I = \int_{-\infty}^{\infty} f(x) dx$.
- (i) By making the transformation $x = 1/t$, show that $I = 2(I_1 + I_2)$ where
- $$I_1 = \int_0^1 \frac{1 - e^{-t^2}}{t^2} dt, \quad I_2 = \int_1^{\infty} \frac{1 - e^{-t^2}}{t^2} dt.$$
- (4 marks)
- (ii) Determine whether the improper integrals I_1 and I_2 exist stating all arguments clearly and concisely. (4 marks)

(Total: 20 marks)

2. (a) A wire is bent into a circle of radius a and stuck on a *weightless* plane. Using polar coordinates, the wire has density per unit length given by $\rho(\theta)$, $0 \leq \theta < 2\pi$. You can also assume that the wire's cross-section is small so that all the mass is concentrated on the curve $r = a$. The centre of mass of the system is at the point C which has Cartesian coordinates (X, Y) . Find the total mass M of the wire and show using moments that

$$X = \frac{1}{M} \int_0^{2\pi} a^2 \cos \theta \rho(\theta) d\theta, \quad Y = \frac{1}{M} \int_0^{2\pi} a^2 \sin \theta \rho(\theta) d\theta.$$

(3 marks)

- (b) If $\rho(\theta) = \rho_0$ is constant, use the results above to find the centre of mass C . Explain your result physically. (2 marks)

- (c) Here, the wire in part (b) is now modified as follows. An identical wire of length $2\theta_0 a$, $0 \leq \theta_0 \leq \pi$, is bent into a circular arc of radius a and stuck on the existing wire.

- (i) What is the centre of mass when $\theta_0 = 0$ and $\theta_0 = \pi$? Explain physically. (2 marks)
- (ii) When $0 < \theta_0 < \pi$ it is found that C is at a radial distance X from the centre $(0, 0)$. Find a formula for X as a function of θ_0 . (3 marks)
- (iii) Use your result in (c)(ii) to show that maximum radial position of the centre of mass is attained for an added wire of length $2a\theta^*$ where θ^* satisfies $\tan \theta^* = \pi + \theta^*$. Prove that $0 < \theta^* < \pi/2$ using a graphical or analytical argument. (3 marks)

- (d) Revisit part (c) above, but now stack N wires of length $2\theta_0 a$ on the existing wire - you may assume that the wires are stacked on top of each other and that their cross-sectional radius can be ignored in your calculations (each one still has mass per unit length equal to ρ_0).

- (i) Show that for a given θ_0 the radial position of the centre of mass is given by

$$X = \frac{aN \sin \theta_0}{\pi + N\theta_0}. \quad (1)$$

(3 marks)

- (ii) Find an equation for the angle θ_0 that will maximise X found in equation (1) above. Solve it approximately for N large, and show that in this case the maximal X is given by

$$X = a \left[1 - \frac{1}{2!} \left(\frac{3\pi}{N} \right)^{2/3} + \dots \right]. \quad (2)$$

Do the large N results make sense intuitively? Explain. (4 marks)

(Total: 20 marks)

3. (a) (i) Consider the function $\cos x$ over the interval $0 \leq x \leq \pi/2$. Write down its two-term Taylor series of about $x = \pi/2$. Your expansion should include a leading order term and a remainder. State clearly any variables you may have introduced. (2 marks)
- (ii) Now consider the function $f(x) = 1/\sqrt{\cos x}$ in the interval $\pi/4 \leq x < \pi/2$. Use your calculations in 3(a)(i) above to show that

$$f(x) \leq \frac{1}{\sqrt{\frac{\pi}{2} - x}}.$$

(3 marks)

- (iii) Next, consider the function $g(x) = 1/\sqrt{\cos x}$, $0 \leq x \leq \pi/4$. Write down a two-term Taylor series about $x = 0$ (including the remainder as in part (i) above), and use it to show that

$$g(x) \leq 1 + \frac{5}{2^{11/4}} x^2.$$

(3 marks)

- (iv) Consider the improper integral

$$\mathcal{I} = \int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}}.$$

Establish whether $\mathcal{I}$ exists or not. If it does not, explain why. If it does, find a bound for it. (4 marks)

- (b) The function

$$h(x) = \begin{cases} \frac{\pi}{2}, & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

is extended on the whole of the real line as an *even* 2π -periodic function.

- (i) Find the Fourier series of $h(x)$. (4 marks)
- (ii) Use your results in (3)(b)(i) to write down, with justification, the Fourier series of

$$F(x) = \begin{cases} 0, & 0 \leq x < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < x < \pi \end{cases}$$

extended on the whole of the real line as an *odd* 2π -periodic function. What values does the Fourier series of $F(x)$ converge to at $x = \pm\pi$ and $x = \pm\frac{\pi}{2}$? (4 marks)

(Total: 20 marks)

4. (a) Consider the following linear ODE:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 12t^2e^{-t}H(2-t),$$

where H denotes the Heaviside function.

- (i) Find the general solution to the corresponding homogenous ODE using an appropriate basis for the solution space and prove the linear independence of the basis function chosen. (4 marks)
- (ii) Find the solution to the full non-homogenous ODE given initial condition of

$$y(-1) = \frac{dy}{dt}(-1) = e.$$

(7 marks)

- (b) Consider the function $f(x)$ defined as

$$f(x) = \begin{cases} (1-x^2) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

- (i) Obtain the Fourier cosine transform of $f(x)$. (4 marks)
- (ii) Use your results in part (i) to show

$$\int_0^\infty \left(\frac{\sin u - u \cos u}{u^3} \right)^2 du = \frac{\pi}{15}.$$

(2 marks)

- (iii) Obtain the Fourier transform of

$$g(x) = \frac{\sin 2x - 2x \cos 2x}{x^3}.$$

(3 marks)

(Total: 20 marks)

5. (a) Consider the following first order nonlinear ODE:

$$\frac{dx}{dt} = \frac{2\alpha x^2 - x}{(\alpha + x)^2} - x$$

where $\alpha \in \mathbb{R}$ is a parameter.

- (i) Find the fixed and singular points of the system and determine the stability of the fixed points for representative values of α using sketches of appropriate graphs. You can use the help of computer plotting tools (such as www.desmos.com). (7 marks)
- (ii) Sketch the bifurcation diagram of this ODE and classify any bifurcations found. Is there bifurcation at $\alpha = 0$, why? (4 marks)
- (iii) Using the bifurcation diagram find the asymptotic behaviour of $x(t)$ for $\alpha = 0.5$ and $x(0) < -1$. (2 marks)

- (b) Consider the following ODE

$$(1 - y^2 e^{-y} \sin x) dx + (x + 2y e^{-y} \cos x) dy = 0$$

- (i) Determine if the equation is exact. (2 marks)
- (ii) Solve the ODE either directly or with the aid of a suitable integrating factor. (5 marks)

(Total: 20 marks)

6. (a) Consider the following system of linear ODEs:

$$\begin{aligned}\frac{dx}{dt} &= -3x + y \\ \frac{dy}{dt} &= -6x + 2y\end{aligned}$$

- (i) Find the general solution of this system. Describe the fixed point(s) of the system and their stability. (6 marks)
 - (ii) Sketch the phase portrait for this system in the (x, y) plane. (3 marks)
 - (iii) Find the particular solution when $x(0) = 0$ and $y(0) = 1$ and draw the corresponding trajectory in the phase plane. What is the asymptotic behaviour. (2 marks)
- (b) Consider function $z(x, y)$ given in implicit form

$$x(y^2 + x^2) = ze^{(yz)}.$$

- (i) Estimate maximum uncertainty in z if $x = 1 \pm 0.01$ and $y = 0 \pm 0.02$. (5 marks)
- (ii) Evaluate

$$\left(\frac{\partial^2 z}{\partial x^2} \right)_y$$

(4 marks)

(Total: 20 marks)

1. (a) (i) Calculate $f' = -\frac{2}{x^3}e^{-1/x^2}$, $f'' = \frac{6}{x^4}e^{-1/x^2} - \frac{4}{x^6}e^{-1/x^2}$. We have $\lim_{x \rightarrow 0} e^{-1/x^2} = 0$, and $\lim_{x \rightarrow 0} \frac{1}{x^p} e^{-1/x^2} = 0$ for all p . The latter is shown by taking logs

sim. seen ↓

$$\log \left[\frac{e^{-1/x^2}}{x^n} \right] = -\frac{1}{x^2} - n \log x \rightarrow -\infty \text{ as } x \rightarrow 0.$$

Hence $\lim_{x \rightarrow 0} f(x) = 1$, $\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} f''(x) = 0$.

Hence, define $f(0) = 1$, $f'(0) = f''(0) = 0$ to get the desired continuity.

4, A

- (ii) $f^{(n)}(x)$ contains terms of the form $\frac{e^{-1/x^2}}{x^p}$ where p is a positive integer. By the result above we have $\lim_{x \rightarrow 0} f^{(n)}(x) = 0$.

unseen ↓

2, B

- (b) (i) Inflection points satisfy $f'' = \frac{6}{x^4}e^{-1/x^2} - \frac{4}{x^6}e^{-1/x^2} = 0$. For $x \neq 0$ we have $x = \pm(2/3)^{1/2}$.

sim. seen ↓

Since $\lim_{x \rightarrow \pm\infty} (1 - e^{-1/x^2}) = 0$, we have $0 < f(x) \leq 1$.

3, A

- (ii) A sketch is given in Figure 1. Inflection points are at $(\pm(2/3)^{1/2}, 1 - \exp(-3/2))$.

unseen ↓

3, B

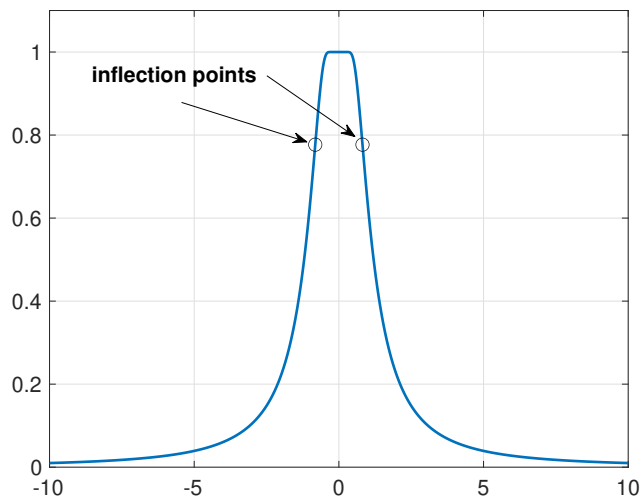


Figure 1: Figure for problem 1(b)(ii).

- (c) (i) If $x = 1/t$ then $dx = -1/t^2$. Also due to symmetry ($f(-x) = f(x)$)

sim. seen ↓

$$\begin{aligned} \int_{-\infty}^{\infty} (1 - e^{-1/x^2}) dx &= 2 \int_0^{\infty} (1 - e^{-1/x^2}) dx \\ &= 2 \int_{\infty}^0 (1 - e^{-t^2}) (-1/t^2) dt = 2 \int_0^{\infty} \frac{1 - e^{-t^2}}{t^2} dt \\ &= 2 \int_0^1 \frac{1 - e^{-t^2}}{t^2} dt + 2 \int_1^{\infty} \frac{1 - e^{-t^2}}{t^2} dt = 2(I_1 + I_2). \end{aligned}$$

1, A

3, C

- (ii) Begin with I_1 . The integrand is defined and bounded for $0 < t \leq 1$. At $t = 0$ calculate (I used L'Hopital's rule but Taylor expansions also fine)

sim. seen ↓

$$\lim_{t \rightarrow 0} \frac{1 - e^{-t^2}}{t^2} = \lim_{t \rightarrow 0} \frac{2te^{-t^2}}{2t} = 1.$$

Hence, we have a bounded integrand on a bounded domain implying $0 < I_1 \leq M_1$ for some constant M_1 .

2, D

Next consider I_2 . There are different ways to do this, but the most straightforward is

$$I_2 < \int_1^\infty \frac{1}{t^2} dt = 1.$$

2, D

2. (a) Using Riemann sums (or directly), an element of the wire has length $a d\theta$ hence mass $a\rho(\theta)d\theta$. Adding all these pieces and taking the limit gives

meth seen ↓

1, A

$$M = \int_0^{2\pi} a\rho(\theta)d\theta.$$

Take moments about the y -axis.

Moment of the element $a\rho(\theta)d\theta$ is $a\rho(\theta)d\theta \times a \cos \theta$.

Summing over all elements and balancing with the moment of the total mass gives

1, A

$$MX = \int_0^{2\pi} a^2 \cos \theta \rho(\theta) d\theta.$$

Analogously, moments about the x -axis provide the moment $a\rho(\theta)d\theta \times a \sin \theta$ for the element and hence

1, A

$$MY = \int_0^{2\pi} a^2 \sin \theta \rho(\theta) d\theta.$$

meth seen ↓

- (b) If $\rho(\theta) = \rho_0$, then the integrals become proportional to $\int_0^{2\pi} \cos \theta d\theta$ and $\int_0^{2\pi} \sin \theta d\theta$, hence $X = Y = 0$. This is not surprising since due to symmetry the centre of mass is the centre of the circle.

2, A

- (c) (i) When $\theta_0 = 0$ we have a uniform density ρ_0 hence $X = Y = 0$. When $\theta_0 = \pi$ we have two wires, one on top of each other, hence the density is again uniform but equal to $2\rho_0$, hence $X = Y = 0$.

meth seen ↓

2, B

- (ii) First find the total mass of the *composite* wire. This is

unseen ↓

$$M = 2\pi a\rho_0 + 2\theta_0 a\rho_0 = 2a\rho_0(\pi + \theta_0).$$

By symmetry the centre of mass can be taken to be at $(X, 0)$. Moments about the y -axis gives

3, C

$$\begin{aligned} MX &= \int_{-\theta_0}^{\theta_0} (2\rho_0)(a d\theta)(a \cos \theta) + \int_{\theta_0}^{2\pi-\theta_0} (\rho_0)(a d\theta)(a \cos \theta) \\ &= 2a^2\rho_0 [\sin \theta]_{-\theta_0}^{\theta_0} + a^2\rho_0 [\sin \theta]_{\theta_0}^{2\pi-\theta_0} \\ &= 2a^2\rho_0 2 \sin \theta_0 + a^2\rho_0 (-2 \sin \theta_0) = 2a^2\rho_0 \sin \theta_0 \\ &\Rightarrow \\ X &= \frac{a \sin \theta_0}{\pi + \theta_0}. \end{aligned}$$

sim. seen ↓

- (iii) Need to maximise X above (the minimum is 0 by inspection). Differentiate and set derivative to 0.

$$\begin{aligned} \frac{dX}{d\theta_0} &= \frac{a \cos \theta_0}{\pi + \theta_0} - \frac{a \sin \theta_0}{(\pi + \theta_0)^2} = 0 \Rightarrow \\ \tan \theta^* &= \pi + \theta^* \quad \text{as required.} \end{aligned}$$

Sketch the curves $\tan \theta^*$ and $\pi + \tan \theta^*$ for $0 \leq \theta^* < \pi/2$, on the same plot and consider their point of intersection. Latter line has slope 1 and passes through $(0, \pi)$. Former curve starts at $(0, 0)$ and tends to $+\infty$ as $\theta^* \rightarrow \pi/2$. Hence they intersect at $0 < \theta^* < \pi/2$. See Figure 2.

No marks given for anything less analytic than this, e.g. numerical evaluations.

3, B

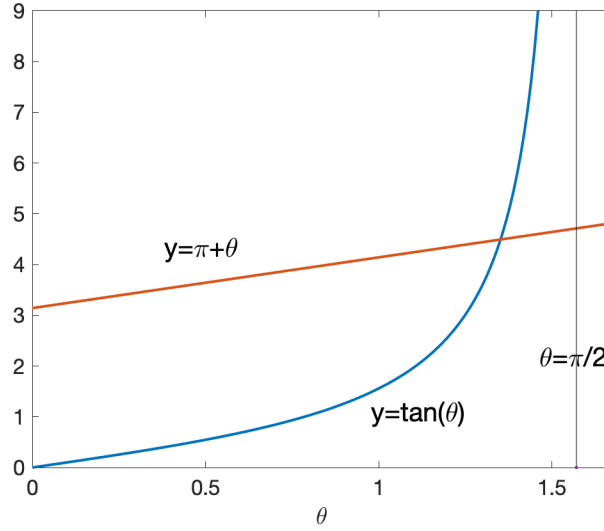


Figure 2: Figure for problem 2(c)(iii).

- (d) (i) Work as in part (b)(ii) modifying the number of wires present. Find

$$M = 2\pi a \rho_0 + 2\theta_0 a \rho_0 N,$$

and

$$MX = \int_{-\theta_0}^{\theta_0} (N+1) \rho_0 a^2 \cos \theta d\theta + a^2 \rho_0 (-2 \sin \theta_0) = 2a^2 \rho_0 N \sin \theta_0,$$

hence

$$X = \frac{aN \sin \theta_0}{\pi + N\theta_0}. \quad (1)$$

- (ii) As before, differentiate wrt θ_0 and look for a local maximum. Find

$$\frac{dX}{d\theta_0} = \frac{aN \cos \theta_0}{\pi + N\theta_0} - \frac{aN^2 \sin \theta_0}{(\pi + N\theta_0)^2} = 0 \Rightarrow \tan \theta^* = \frac{\pi}{N} + \theta^*.$$

From part (d)(ii) we have,

$$X = \frac{a \sin \theta^*}{\frac{\pi}{N} + \theta^*} = a \cos \theta^*. \quad (2)$$

Since $\pi/N \ll 1$ we can solve for θ^* using a Taylor expansion of (1),

$$\theta^* + \frac{1}{3}(\theta^*)^3 + \dots = \theta^* + \frac{\pi}{N} \Rightarrow \theta^* = \left(\frac{3\pi}{N}\right)^{1/3} + \dots \ll 1.$$

Putting this into (2) and Taylor expanding for small θ^* , we find

$$X = a \left[1 - \frac{1}{2!} \left(\frac{3\pi}{N}\right)^{2/3} + \dots \right]$$

For N large the centre of mass moves to the edge of the circle. Makes sense since a lot of mass is concentrated in a small arc and so the centre of mass tends towards the centre of mass of the added wire sectors - the rest of the wire is less relevant.

3. (a) (i) Taylor's theorem at $x = \pi/2$ gives

seen ↓

$$\begin{aligned}\cos x &= \cos(\pi/2 + x - \pi/2) = \\ &= \cos(\pi/2) + (x - \pi/2)(-\sin(\pi/2)) + \frac{1}{2!}(x - \pi/2)^2(-\cos(\pi/2)) + \frac{1}{3!}(x - \pi/2)^3(+\sin \xi) \\ &= \left(\frac{\pi}{2} - x\right) - \frac{1}{3!}\left(\frac{\pi}{2} - x\right)^3 \sin \xi,\end{aligned}\quad (3)$$

where ξ is a number between $\pi/2 - x$ and $\pi/2$.

2, A

- (ii) Using the result above we have - and noting that the interval is $\pi/4 \leq x < \pi/2$:

meth seen ↓

$$\begin{aligned}f(x) &= \frac{1}{\sqrt{\cos x}} = \frac{1}{\sqrt{\frac{\pi}{2} - x}} \frac{1}{\left[1 - \frac{1}{3!}\left(\frac{\pi}{2} - x\right)^2 \sin \xi\right]^{1/2}} \\ &\geq \frac{1}{\sqrt{\frac{\pi}{2} - x}},\end{aligned}$$

since $0 < \frac{1}{3!}\left(\frac{\pi}{2} - x\right)^2 \sin \xi < 1$.

3, B

- (iii) Need the first two *non-zero* derivatives $g^{(n)}(x)$, $n = 0, 1, 2, \dots$, when evaluated at $x = 0$. Calculate:

unseen ↓

$$\begin{aligned}g(x) &= (\cos x)^{-1/2}, & g(0) &= 1, \\ g'(x) &= \frac{1}{2}(\cos x)^{-3/2} \sin x, & g'(0) &= 0, \\ g''(x) &= \frac{3}{4}(\cos x)^{-5/2} \sin^2 x + \frac{1}{2}(\cos x)^{-1/2}, & g''(0) &\neq 0.\end{aligned}$$

The two terms Taylor series is

$$g(x) = 1 + \frac{1}{2!}x^2 g''(\eta) = 1 + \frac{1}{2!}x^2 \frac{[1 + (3/2) \tan^2 \eta]}{2(\cos \eta)^{1/2}}$$

where $0 \leq \eta \leq \pi/4$. Using the bounds for η , allows the estimate

$$g(x) \leq 1 + x^2 \frac{1}{2!} \frac{[1 + (3/2) \tan^2 \pi/4]}{2(\cos \pi/4)^{1/2}} = 1 + x^2 \frac{5}{2^{11/4}}$$

3, C

- (iv) Write the improper integral over $[0, \pi/4]$ and $[\pi/4, \pi/2)$ and use the estimates for $f(x)$, $g(x)$ found above:

sim. seen ↓

$$\begin{aligned}\mathcal{I} &= \int_0^{\pi/4} \frac{dx}{\sqrt{\cos x}} + \int_{\pi/4}^{\pi/2} \frac{dx}{\sqrt{\cos x}} \\ &\leq \int_0^{\pi/4} \left(1 + x^2 \frac{5}{2^{11/4}}\right) dx + \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{\cos x}} dx \\ &= \frac{\pi}{4} + \frac{(\pi/4)^3}{3} \frac{5}{2^{11/4}} + \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{\cos x}} dx.\end{aligned}$$

From part 3(a)i we have that near $x = \pi/2$, $1/\sqrt{\cos x} \approx 1/\sqrt{\pi/2 - x}$. This is singular at $x = \pi/2$ but integrable. For $\pi/4 \leq x \leq \pi/2$ we have, therefore $1/\sqrt{\cos x} < A/\sqrt{\pi/2 - x}$ where $A > 0$ is chosen to be big enough, e.g. $A = 10$ certainly works but more analysis can make the choice better. This implies $\int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{\cos x}} dx < 10 \int_{\pi/4}^{\pi/2} (1/\sqrt{\pi/2 - x}) dx = 20(\pi/4)^{1/2} < \infty$, hence bounded.

2, B

2, D

(b) (i) Since the function is even and 2π -periodic, we have

seen ↓

$$h(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \quad a_n = \frac{2}{\pi} \int_0^{\pi/2} \frac{\pi}{2} \cos nx \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \cos nx \, dx.$$

Hence

4, A

$$\begin{aligned} a_0 &= \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{2} \\ a_n &= \frac{1}{n} \sin(n\pi/2) + \frac{2}{\pi} \left(\left[(\pi - x) \frac{\sin nx}{n} \right]_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \frac{\sin nx}{n} dx \right) \\ &= \frac{2}{\pi n^2} (\cos(n\pi/2) - \cos n\pi). \end{aligned}$$

(ii) $h(x)$ is continuous, hence the derivative of its Fourier series will converge to the piecewise derivative of the function, i.e. $F(x)$. The Fourier series of $F(x)$ is

sim. seen ↓

$$F(x) = - \sum_{n=1}^{\infty} n a_n \sin nx,$$

with a_n as above.

2, A

At $x = \pm\pi/2$ F converges to $\mp 1/2$, and at $x = \pm\pi$ F converges to 0.

2, D

Review of mark distribution:

Total A marks: 24 of 32 marks

Total B marks: 15 of 20 marks

Total C marks: 9 of 12 marks

Total D marks: 12 of 16 marks

Total marks: 60 of 80 marks

Total Mastery marks: 0 of 20 marks

4 (a (i)) We try $y = e^{\lambda t}$ and we obtain

seen ↓

$$\lambda^2 + 2\lambda + 1 = 0$$

which has a repeated root of $\lambda = -1$, so as the basis for the solution space we can use $\{e^{\lambda t}, te^{\lambda t}\}$. To check linear independence we calculate the Wronskian which is in this case:

$$W = \det \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ \lambda e^{\lambda t} & e^{\lambda t} + \lambda te^{\lambda t} \end{bmatrix} = e^{2\lambda t} \neq 0.$$

So, we have

$$y_{CF} = c_1 e^{-t} + c_2 t e^{-t}.$$

4, A

(a (ii)) We consider the ODE for $t < 2$. We try the ansatz

unseen ↓

$$y_{PI} = At^4 e^{-t}.$$

By substituting into the ODE, We obtain $A = 1$, so we have (for $t < 2$):

$$y_{GS} = c_1 e^{-t} + c_2 t e^{-t} + t^4 e^{-t}.$$

Using the initial conditions at $t = -1$, we obtain:

$$c_1 - c_2 + 1 = 1,$$

$$-c_1 + c_2 + c_2 - 4 - 1 = 1.$$

Which gives $c_1 = c_2 = 6$. So the solution for $t < 2$ is:

$$y = 6e^{-t} + 6te^{-t} + t^4 e^{-t}.$$

4, D

For $t > 2$ we have just the homogenous ODE, so using the solution in part (i) we have:

$$y_{GS} = c'_1 e^{-t} + c'_2 t e^{-t}.$$

We can set the new constants of integration (c'_1 and c'_2) for $t > 2$ by requiring y and its derivative to be continuous at $t = 2$. We have:

$$y(2) = (6 + 12 + 16)e^{-2} = (c'_1 + 2c'_2)e^{-2},$$

$$y'(2) = (-6 + 6 - 12 + 32 - 16)e^{-2} = (-c'_1 + c'_2 - 2c'_2)e^{-2}.$$

So, we obtain $c'_1 = -42$ and $c'_2 = 38$. So we have for $t > 2$:

$$y = -42e^{-t} + 38te^{-t}.$$

3, C

(b (i))

sim. seen ↓

$$\hat{f}_e(\omega) = \frac{1}{2}\hat{f}(\omega) = \frac{1}{2} \int_1^1 e^{-i\omega x}(1-x^2) dx = \int_0^1 \cos(\omega x)(1-x^2) dx = 2 \left(\frac{\sin \omega}{\omega^3} - \frac{\cos \omega}{\omega^2} \right).$$

4, A

(b (ii)) Using the results above and the energy theorem we obtain:

$$\frac{16}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \omega - \omega \cos \omega}{\omega^3} \right)^2 d\omega = \int_{-1}^1 (1 - x^2)^2 dx = \frac{16}{15}.$$

sim. seen ↓

Which gives us the results requested.

2, B

(b (iii)) From the result in part b (i) we see that $\mathcal{F}\{f(x)\} = \frac{1}{2}g(\frac{\omega}{2})$. Using symmetry formula and also the fact f is an even function, we have:

sim. seen ↓

$$\mathcal{F}\left\{\frac{1}{2}g\left(\frac{x}{2}\right)\right\} = 2\pi f(-\omega) \Rightarrow \mathcal{F}\left\{g\left(\frac{x}{2}\right)\right\} = 4\pi f(\omega).$$

Now using the second basic property of Fourier transforms from the lecture notes (section 2.1), we get:

$$\mathcal{F}\left\{g\left(\frac{x}{2}\right)\right\} = 2\hat{g}(2\omega) = 4\pi f(\omega) \Rightarrow \hat{g}(\omega) = 2\pi f\left(\frac{\omega}{2}\right).$$

3, B

- 5 (a (i)) Singular point is $x_{sing} = -\alpha$ and the fixed points can be obtained by equating the LHS to zero, which only gives on real fixed point:

meth seen ↓

$$x_1^* = 0.$$

4, A

The stability of these points can be seen in Figure 1.

3, C

- (a (ii)) The bifurcation diagram can be seen in Figure 2. There are no bifurcation at $\alpha = 0$ or any other point, since the number and stability of fixed points are not changing.

unseen ↓

4, D

- (a (iii)) x approaches to $x_{sing} = -0.5$ in finite time and the equation becomes undefined beyond this (if the starting point was $x > -0.5$ the solution would asymptotically approach $x = 0$).

meth seen ↓

2, A

- (b (i)) Consider the ODE as $F(x, y)dx + G(x, y)dy = 0$. We see that it is not exact as

meth seen ↓

$$\frac{\partial F}{\partial y} = (y^2 - 2y) \sin x e^{-y},$$

$$\frac{\partial G}{\partial x} = 1 - 2ye^{-y} \sin x.$$

are not equal.

2, A

- (b (ii)) We try $\lambda(y)$ and obtain:

meth seen ↓

$$\frac{d\lambda}{dy} F + \lambda \frac{\partial F}{\partial y} = \lambda \frac{\partial G}{\partial x}$$

which gives the following ODE.

$$\frac{d\lambda}{dy} = \lambda \quad \Rightarrow \quad \lambda = e^y.$$

Using this we have a $u(x, y)$:

$$\frac{\partial u}{\partial x} = \lambda F = e^y - y^2 \sin x \quad \Rightarrow \quad u = e^y x + y^2 \cos x + h(y).$$

Using $G\lambda = \frac{\partial u}{\partial y}$, we obtain $h(y) = c$, so we have the implicit solution:

$$u(x, y) = e^y x + y^2 \cos x + c = 0.$$

5, B

6 . (a)

meth seen ↓

$$A = \begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 0, \quad \lambda_2 = -1.$$

We can obtain the corresponding eigenvectors.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So the solution in the vector form is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}.$$

4, A

There is a line of fixed points along $\vec{v}_1$ and these are stable as $\lambda_2 < 0$.

2, A

(a (ii)) The phase portrait can be seen in Figure 3.

meth seen ↓

3, B

(a (iii))

$$c_1 + c_2 = 0,$$

$$3c_1 + c_2 = 1.$$

So we obtain $c_1 = 1$ and $c_2 = -1$. The trajectory is shown in Figure 3.

2, B

(b (i))

meth seen ↓

$$f = x(y^2 + x^2) - ze^{(yz)} = 0.$$

We have $z(1, 0) = 1$. For the partial derivatives using implicit differentiation, we get:

$$\left(\frac{\partial z}{\partial x} \right)_y = - \frac{\left(\frac{\partial f}{\partial x} \right)_{y,z}}{\left(\frac{\partial f}{\partial z} \right)_{x,y}} = \frac{y^2 + 3x^2}{(zy + 1)e^{yz}} = 3.$$

$$\left(\frac{\partial z}{\partial y} \right)_x = - \frac{\left(\frac{\partial f}{\partial y} \right)_{x,z}}{\left(\frac{\partial f}{\partial z} \right)_{x,y}} = \frac{2xy - z^2 e^{yz}}{(zy + 1)e^{yz}} = -1.$$

3, C

So we have

$$dz = \left| \left(\frac{\partial z}{\partial x} \right)_y dx \right| + \left| \left(\frac{\partial z}{\partial y} \right)_x dy \right| = 3 \times 0.01 + 1 \times 0.02 = 0.05$$

2, A

- (b (ii)) To evaluate this second partial derivative we note that the first partial derivative is given as a function $g(x, y, z(x, y))$ and we have

$$\left(\frac{\partial g}{\partial x}\right)_y = \left(\frac{\partial g}{\partial x}\right)_{y,z} + \left(\frac{\partial g}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial x}\right)_y.$$

So using this result and the result in part (i) we obtain:

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_y = \frac{6x}{(zy+1)e^{yz}} - \frac{(y^2+3x^2)^2}{(zy+1)^3 e^{3yz}} (ye^{yz} + (zy+1)ye^{yz}).$$

Review of mark distribution:

Total A marks: 24 of 32 marks

Total B marks: 15 of 20 marks

Total C marks: 9 of 12 marks

Total D marks: 12 of 16 marks

Total marks: 60 of 80 marks

Total Mastery marks: 0 of 20 marks

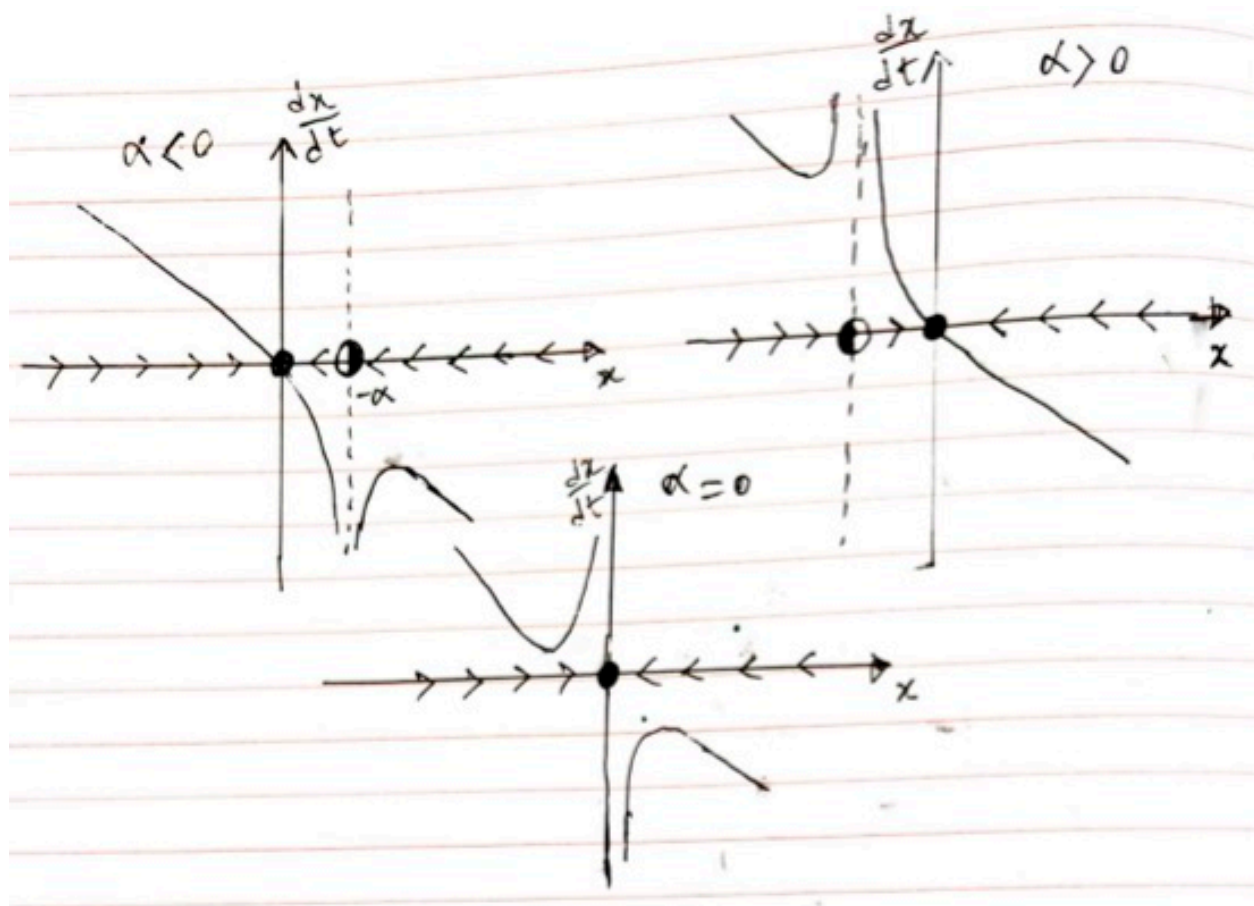


Figure 1: Stability of fixed point for question 5a(i).

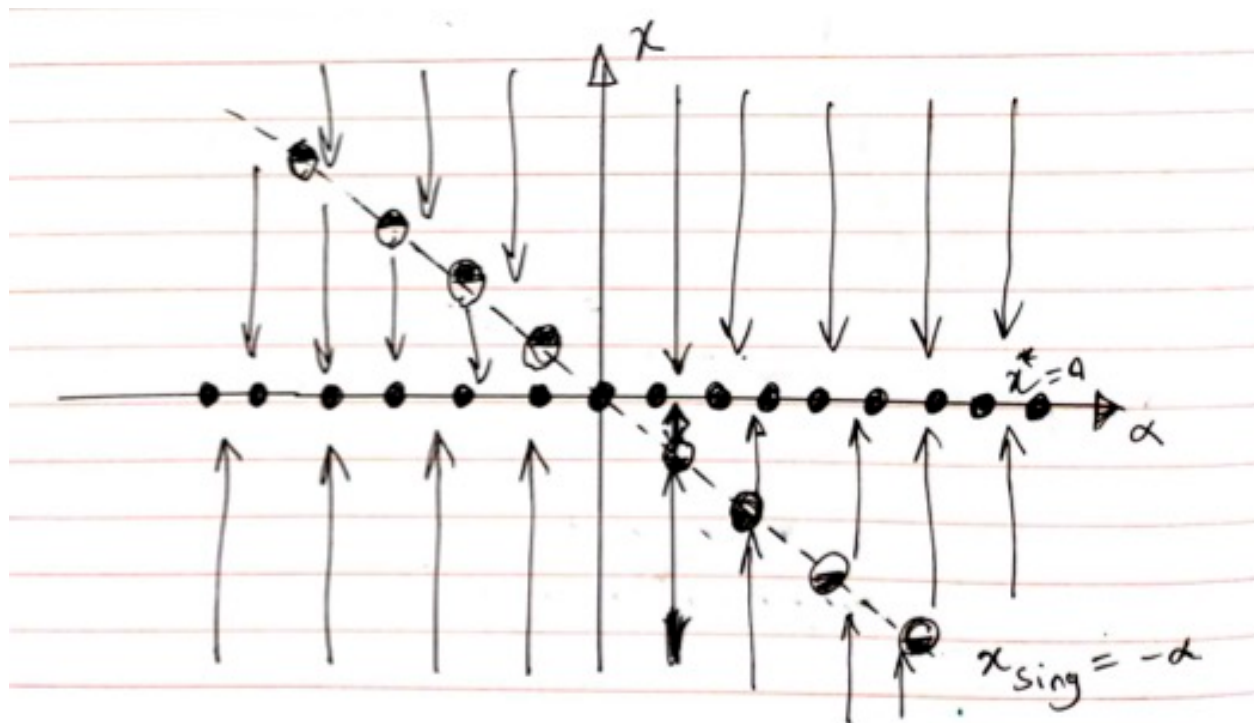


Figure 2: Bifurcation diagram for question 5a(ii).

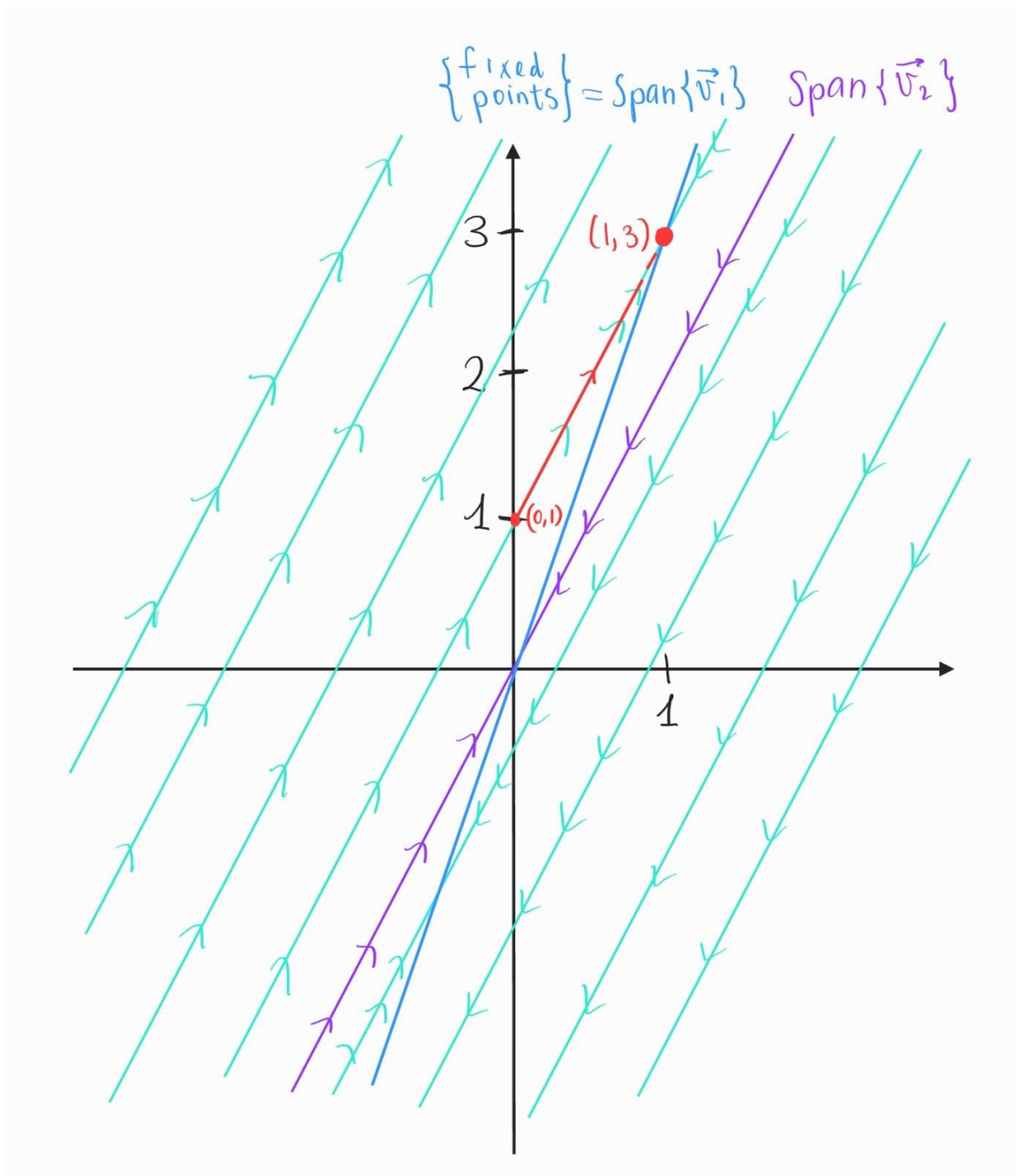


Figure 3: Phase portrait for question 6a(ii) and (iii).

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please

ExamModuleCode	Question	Comments for Students
		The question was well-balanced and standard, and the students handled it in the average way
Calculus and Applications_MATH40004	1	The majority of students managed to do parts (a), (b) and c(i), which are relatively simple. Unfortunately, quite a few were unable to derive the formulae for the centre of mass as required in (a). The remaining parts of the question were found to be challenging, and the performance turned out to be below expectation. In fact, even if one cannot answer c(ii) and d(i), it is still possible to
Calculus and Applications_MATH40004	2	tackle c(iii) and d(ii). The first part on Taylor series could have been done better. It was non-standard in several ways and I felt that students did not spend enough time thinking before calculating. For example, two terms
Calculus and Applications_MATH40004	3	including remainder means exactly that. Unfortunately this was ignored in many cases with 3 or 4a had some routine parts such as finding the solution to the homogenous ODE. However, a lot of students failed to use the continuity at $t = 2$ to set the constants of integration correctly. Lots of students got 4b(i) and (ii) correctly and rarely students got 4b(iii) exactly correct.
Calculus and Applications_MATH40004	4	For part (a) lots of students missed did not include the singularity at $x = -\alpha$ in their bifurcation diagram. This singularity is half-stable, so the asymptotic behavior in part (iii) is towards the singular point, which was again missed by many students. Part (b) was more routine and lots of students got
Calculus and Applications_MATH40004	5	it correctly. Most people did well on part (a). However, few people were able to successfully complete part (b).
Calculus and Applications_MATH40004	6	