

**BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May 2024**

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Calculus and Applications

Date: Tuesday, May 14, 2024

Time: 14:00 – 17:00 (BST)

Time Allowed: 3 hours

This paper has 6 Questions.

Please Answer Each Question in a Separate Answer Booklet

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Allow margins for marking.

Credit will be given for all questions attempted.

Each question carries equal weight.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO

1. (a) Find $\lim_{x \rightarrow 0^+} \frac{e^{-1/x}}{x^p}$ where $p > 0$ is arbitrary. (2 marks)
- (b) Let $f(x) = 1 - e^{-1/|x|}$, $x \neq 0$.
- (i) How should $f(0)$ be defined to make $f(x)$ continuous there. (2 marks)
- (ii) Calculate $f'(x)$ for $x \neq 0$, and determine $\lim_{x \rightarrow 0} f'(x)$. Next, argue that $\lim_{x \rightarrow 0} \frac{d^n f}{dx^n} = 0$. (4 marks)
- (iii) Find all points of inflection of $f(x)$ and sketch the function for $-\infty < x < \infty$. (3 marks)
- (iv) Determine whether the improper integral $\int_0^\infty f(x) dx$ exists. (3 marks)
- (c) Consider next the improper integral $J = \int_1^\infty \left[f(x) - \frac{1}{\sqrt{1+x^2}} \right] dx$, where $f(x)$ is as defined in (b) above.
- (i) By making the substitution $y = 1/x$ re-write J as the transformed improper integral $\tilde{J} = \int_0^1 g(y) dy$ and give $g(y)$. (3 marks)
- (ii) Find $\lim_{y \rightarrow 0} g(y)$ and use your result to determine whether $\tilde{J}$ (and hence J) is bounded. (3 marks)

(Total: 20 marks)

2. (a) Show that $\frac{d}{dx} \log(1/\cos x) = \tan x$, and use this result to find

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \tan \frac{k\pi}{4n}.$$

[Hint: Form an appropriate Riemann sum]

(4 marks)

- (b) (i) Write down the first two non-zero terms of the Taylor expansion of $\tan x$ about $x = 0$. (3 marks)
- (ii) A rod of length $\pi/4$ has non-uniform density given by $\rho(x) = \rho_0 \tan x$. Find the coordinate $\bar{x}$ of the rod's centre of mass measured from the origin $x = 0$; give your answer in terms of $\mathcal{J} = \int_0^{\pi/4} x \tan x dx$. (4 marks)
- (iii) Use the results of part (b)(i)-(ii) above to show that

$$\bar{x} \approx \frac{\pi^3}{96 \log 2} \left(1 + \frac{\pi^2}{80} \right).$$

(3 marks)

- (c) A two-dimensional curve is given by the following parametric equations

$$x(t) = e^{-t} \cos t, \quad y(t) = e^{-t} \sin t, \quad t \geq 0.$$

Sketch the curve for $0 \leq t \leq 2\pi$ in the $x - y$ plane, and discuss what happens as $t \rightarrow \infty$. Find the curve's length as $t \rightarrow \infty$. (6 marks)

(Total: 20 marks)

3. (a) (i) The function $\sin(x/2)$ on the interval $-\pi < x \leq \pi$ is extended periodically to produce a 2π -periodic function. Find its Fourier series.
 [You may use the formulas $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$.] (5 marks)
- (ii) To what values does the Fourier series you just found converge to at $x = 0$, $x = \pi/2$ and $x = \pi$? Justify your answers. (2 marks)
- (iii) Use your results to find S where

$$S = \frac{1}{4 \cdot 1^2 - 1} - \frac{3}{4 \cdot 3^2 - 1} + \frac{5}{4 \cdot 5^2 - 1} - \dots$$

(3 marks)

- (b) (i) State without proof Parseval's theorem for a function $g(x)$ that has been represented by its Fourier series on the interval $-\pi \leq x \leq \pi$. (2 marks)
- (ii) Let $f(x)$ be a non-zero 2π -periodic function that is at least four times differentiable. In addition assume that $f(x)$ has zero spatial mean. Define

$$E = \int_{-\pi}^{\pi} f(x) [f''(x) + \varepsilon f''''(x)] dx$$

where $\varepsilon \geq 0$ is a constant and primes denote x -derivatives of the function. Show that

$$E = - \int_{-\pi}^{\pi} (f'(x))^2 dx + \varepsilon \int_{-\pi}^{\pi} (f''(x))^2 dx.$$

(2 marks)

- (iii) Use Parseval's theorem to show that

$$\pi^{-1} E = \sum_{n=1}^{\infty} n^2 (\varepsilon n^2 - 1) (a_n^2 + b_n^2),$$

where a_n and b_n are the Fourier coefficients of $f(x)$. (4 marks)

- (iv) For what values of ε can we guarantee that $E > 0$? (2 marks)

(Total: 20 marks)

4. (a) Consider the following system of coupled, linear, first-order, differential equations:

$$\begin{aligned}\frac{dx}{dt} &= -2x + y, \\ \frac{dy}{dt} &= -4x + 3y.\end{aligned}$$

- (i) Find the general solution of this system. (4 marks)
(ii) Sketch the phase portrait for this system in the phase plane and describe its asymptotic behaviour. (4 marks)

- (b) Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 - |x|, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

- (i) Obtain the Fourier cosine transform of $f(x)$. (4 marks)
(ii) Hence show that

$$\int_0^\infty \left(\frac{1 - \cos u}{u^2} \right)^2 du = \frac{\pi}{6}.$$

(4 marks)

- (iii) Determine the Fourier transform of

$$g(x) = \frac{1 - \cos 3x}{x^2}.$$

(4 marks)

(Total: 20 marks)

5. (a) Consider the following first-order, nonlinear ODE

$$\frac{dx}{dt} = x^3 - (r - 1)x,$$

where $r \in \mathbb{R}$ is a parameter.

- (i) Find the fixed points, x^* , of the system and determine their stability for representative values of r . (4 marks)
- (ii) Sketch the bifurcation diagram (the values of x^* against r) and classify any bifurcations found. (4 marks)

- (b) Consider the third-order, Euler-Cauchy ODE

$$x^3 \frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} = 1 + 8 \cosh(\log x), \quad (1)$$

where $x > 0$.

- (i) Show that the substitution $x = e^z$ transforms the Euler-Cauchy ODE (1) into the third-order, linear, differential equation

$$\frac{d^3 y}{dz^3} - 2 \frac{d^2 y}{dz^2} + \frac{dy}{dz} = 1 + 8 \cosh z.$$

(4 marks)

- (ii) Hence, or otherwise, find the general solution of (1). (8 marks)

(Total: 20 marks)

6. (a) Consider the first-order differential equation

$$\left(\frac{6x}{y} + 3x^2\right) \frac{dy}{dx} + (12 + 5xy) = 0. \quad (2)$$

- (i) Show that (2) is not exact. (2 marks)
- (ii) By seeking an integrating factor of the form $\lambda(x, y) = x^m y^n$, or otherwise, find the general solution of (2). (6 marks)

(b) Consider the function

$$u(x, y) = (2y^2 + 2x - y - 1)(2x - y + 1).$$

- (i) Find the set of points (x_*, y_*) that form the zero contour, i.e., $u(x_*, y_*) = 0$. (1 mark)
- (ii) Find the stationary points of $u(x, y)$. By use of the Hessian of $u(x, y)$, classify each stationary point. (7 marks)
- (iii) Sketch a contour plot of $u(x, y)$, clearly indicating the zero contour ($u(x_*, y_*) = 0$) and a few other representative contours. Your sketch should include the positions of the stationary points and indicate regions of positive and negative u . Using your sketch, argue the consistency of the classifications obtained in part (ii) for the stationary points. (4 marks)

(Total: 20 marks)

1. (a) Let $y = \frac{e^{-1/x}}{x^p}$, $x > 0$. Then

sim. seen ↓

$$\log(y) = -\frac{1}{x} - p \log(x) = -\frac{1}{x}(1 + px \log(x)) \rightarrow -\infty \quad \text{as } x \rightarrow 0+,$$

since $1 + px \log x \rightarrow 1$ as $x \rightarrow 0+$. Hence $y \rightarrow 0$ as $x \rightarrow 0+$.

2, B

- (b) (i) Just showed that $e^{-1/x} \rightarrow 0$ as $x \rightarrow 0+$ (special case $p = 0$). Since $f(x)$ is an even function, then $f(x) \rightarrow 1$ as $x \rightarrow 0-$. Hence we need $f(0) = 1$ for the function to be continuous everywhere.

sim. seen ↓

2, A

- (ii) Chain rule gives $f'(x) = -\frac{1}{x^2}e^{-1/x}$ for $x > 0$. Hence by 1(a) with $p = 2$ we have $\lim_{x \rightarrow 0+} f'(x) = 0$, and it follows that $\lim_{x \rightarrow 0-} f'(x) = 0$ also since $f(x)$ is symmetric (and so $f'(-x) = -f'(x)$).

2, A

We have, again for $x > 0$, $f^{(2)}(x) = \frac{2}{x^3}e^{-1/x} - \frac{1}{x^4}e^{-1/x}$, and $f^{(n)}(x)$ will contain a term proportional to $\frac{1}{x^{n+1}}e^{-1/x}$ and another proportional to $\frac{1}{x^{2n}}e^{-1/x}$. Both $\rightarrow 0$ as $x \rightarrow 0+$ and by symmetry we have $\lim_{x \rightarrow 0} f^{(n)}(x) = 0$.

2, B

- (iii) Set $f^{(2)}(x) = 0$ to find $x = 1/2$, inflection points symmetrically placed at $(1/2, 1 - e^{-2})$ and $(-1/2, 1 - e^{-2})$.

1, A

Sketch given in figure 1.

2, B

- (iv) This can be done different ways, the most direct is to note that the function is bounded at $x = 0$ so no issues there, and so all depends on the behaviour for large x . For large x we have $1 - e^{-1/x} = 1 - (1 - \frac{1}{x} + \frac{1}{2x^2} + \dots) = \frac{1}{x} + \dots$, and so the integral does not exist.

3, B

- (c) (i) $y = 1/x$ implies $dx = (-1/y^2)dy$ and so

seen/sim. seen ↓

$$\tilde{J} = \int_1^0 \left[1 - e^{-y} - \frac{y}{\sqrt{1+y^2}} \right] (-1/y^2) dy = \int_0^1 \frac{1}{y^2} \left[1 - e^{-y} - \frac{y}{\sqrt{1+y^2}} \right] dy$$

- (ii) Enough to look at the behaviour of $g(y)$ for small y (clearly $g(1)$ is bounded) using Taylor/binomial expansions to find

3, B

unseen ↓

$$g(y) = \frac{1}{y^2} \left[1 - (1 - y + \frac{y^2}{2} + \dots) - y(1 - \frac{y^2}{2} + \dots) \right] = -\frac{1}{2} + \dots,$$

hence the function is integrable at $y = 0$ and hence $\tilde{J} < \infty$.

3, C

2. (a) Use the chain rule:

$$\frac{d}{dx} \log(1/\cos x) = \frac{d}{dx} (-\log \cos x) = -\frac{1}{\cos x} (-\sin x) = \tan x.$$

meth seen ↓

unseen ↓

Hence $\int_0^{\pi/4} \tan x \, dx = [\log(1/\cos x)]_0^{\pi/4} = \log \sqrt{2} - \log(1) = \frac{1}{2} \log 2.$

2, A

Now write $\int_0^{\pi/4} \tan x \, dx$ as a Riemann sum. Split $\pi/4$ into n equal intervals of length $h = \pi/(4n)$. Write the lower Riemann sum to find

$$\int_0^{\pi/4} \tan x \, dx \approx \frac{\pi}{4n} \sum_{k=0}^{n-1} \tan \frac{k\pi}{4n},$$

and sending $n \rightarrow \infty$ we find $\lim_{n \rightarrow \infty} \frac{\pi}{4n} \sum_{k=0}^{n-1} \tan \frac{k\pi}{4n} = \frac{1}{2} \log 2$, hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \tan \frac{k\pi}{4n} = \frac{2}{\pi} \log 2.$$

2, D

- (b) (i) Can do it using Taylor's theorem directly or using

seen ↓

$$\tan x = \frac{\sin x}{\cos x} = (1 - \frac{x^3}{3!} + \dots)(1 - \frac{x^2}{2!} + \dots)^{-1} = (1 - \frac{x^3}{3!} + \dots)(1 + \frac{x^2}{2!} + \dots) = x + \frac{x^3}{3} + \dots$$

- (ii) Taking moments about $x = 0$ we find

3, A

$$\bar{x} \int_0^{\pi/4} \rho_0 \tan x \, dx = \int_0^{\pi/4} x \rho_0 \tan x \, dx,$$

sim. seen ↓

4, A

hence

$$\bar{x} = \frac{2}{\log 2} \mathcal{J}.$$

- (iii) A two terms Taylor expansion gives

sim. seen ↓

$$\begin{aligned} \bar{x} &= \frac{2}{\log 2} \int_0^{\pi/4} x(x + \frac{x^3}{3} + \dots) \, dx = \frac{2}{\log 2} \left[\frac{x^3}{3} + \frac{x^5}{15} + \dots \right]_0^{\pi/4} \\ &\approx \frac{2}{\log 2} \frac{\pi^3}{4^3} \frac{1}{3} \left[1 + \frac{\pi^2}{80} \right] = \frac{\pi^3}{96 \log 2} \left[1 + \frac{\pi^2}{80} \right]. \end{aligned}$$

3, A

- (c) In polar coordinates (r, θ) we have $r^2 = x^2 + y^2 = e^{-2t}$ and $\tan \theta = \frac{y(t)}{x(t)} = \tan t$. Hence the curve is a spiral with $\theta = t$ in polar coordinates and $r = e^{-t}$. A sketch is given in figure 2. As $t \rightarrow \infty$ we have $r \rightarrow 0$ so the curve tends to the origin after an infinite number of turns.

sim. seen ↓

unseen ↓

2, B

1, D

To find the length of the curve we can use the parametric element $ds = \sqrt{dx^2 + dy^2} = \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$, hence the length, if it exists, is given by the improper integral

$$L = \int_0^\infty \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^\infty \sqrt{2} e^{-t} dt = \sqrt{2},$$

where the calculation $(dx/dt)^2 + (dy/dt)^2 = (-e^{-t} \sin t - e^{-t} \cos t)^2 + (e^{-t} \cos t - e^{-t} \sin t)^2 = 2e^{-2t}$ has been used.

3, D

[Alternatively, the students may use the polar form for the length $\int_0^\infty \sqrt{(dr/d\theta)^2 + r^2} d\theta$ to obtain the same result.]

3. (a) (i) The function is odd about $x = 0$ so a sine series is appropriate, $\sin(x/2) = \sum_{n=1}^{\infty} b_n \sin nx$, where

seen/sim. seen ↓

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin(x/2) \sin nx \, dx.$$

2, A

From the given formulas we have $\sin nx \sin(x/2) = \frac{1}{2} \cos[(n - 1/2)x] - \frac{1}{2} \cos[(n + 1/2)x]$, hence

2, A

$$\begin{aligned} b_n &= \frac{2}{\pi} \left[\frac{1}{2} \frac{\sin[(n - \frac{1}{2})x]}{n - \frac{1}{2}} - \frac{1}{2} \frac{\sin[(n + \frac{1}{2})x]}{n + \frac{1}{2}} \right]_0^{\pi} = \frac{1}{\pi} \left[\frac{\sin(n - \frac{1}{2})\pi}{n - \frac{1}{2}} - \frac{\sin(n + \frac{1}{2})\pi}{n + \frac{1}{2}} \right] \\ &= \frac{1}{\pi} \left[\frac{\sin n\pi \cos(\pi/2) - \cos n\pi \sin(\pi/2)}{n - \frac{1}{2}} - \frac{\sin n\pi \cos(\pi/2) + \cos n\pi \sin(\pi/2)}{n + \frac{1}{2}} \right] \\ &= \frac{1}{\pi} \left[-\frac{\cos n\pi}{n - \frac{1}{2}} - \frac{\cos n\pi}{n + \frac{1}{2}} \right] = -\frac{\cos n\pi}{\pi} \frac{2n}{n^2 - \frac{1}{4}} = -\frac{8n \cos n\pi}{\pi(4n^2 - 1)} = \frac{(-1)^{n+1} 8n}{\pi(4n^2 - 1)} \end{aligned}$$

seen ↓

- (ii) At points of discontinuity the Fourier series converges to the average value.

• At $x = 0$ series converges to 0.

1, A

• At $x = \pi/2$ series converges to $\sin(\pi/4) = 1/\sqrt{2}$.

1, B

• At $x = \pi$ series converges to 0 since the function is discontinuous and converges to 1 from below and -1 from above.

unseen ↓

- (iii) Put $x = \pi/2$ in the Fourier series just found. Hence

1, C

$$\frac{1}{\sqrt{2}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 8n}{\pi(4n^2 - 1)} \sin(n\pi/2) = \frac{8}{\pi} \left[\frac{1}{4 \cdot 1^2 - 1} - \frac{3}{4 \cdot 3^2 - 1} + \frac{5}{4 \cdot 5^2 - 1} + \dots \right]$$

2, D

hence

$$S = \frac{\pi}{8\sqrt{2}}.$$

seen ↓

- (b) (i) Parseval's theorem states that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} g^2(x) \, dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

where a_n and b_n are the Fourier cosine and sine series coefficients in our usual notation.

2, A

- (ii) Integrate by parts and use periodicity and the given regularity

unseen ↓

$$\begin{aligned} E &= \int_{-\pi}^{\pi} f(f'' + \varepsilon f''') \, dx = [ff' + \varepsilon ff''']_{-\pi}^{\pi} - \int_{-\pi}^{\pi} [(f')^2 + \varepsilon f' f'''] \, dx \\ &= - \int_{-\pi}^{\pi} (f')^2 \, dx - [\varepsilon f' f'']_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \varepsilon (f'')^2 \, dx = - \int_{-\pi}^{\pi} (f')^2 \, dx + \int_{-\pi}^{\pi} \varepsilon (f'')^2 \, dx \end{aligned}$$

as required.

2, C

- (iii) Apply Parseval's theorem to E found above (there is no a_0 term since we have zero mean). Note that the Fourier series of f' is given by $f' = \sum_{n=1}^{\infty} [-na_n \sin nx + nb_n \cos nx]$ and $f'' = \sum_{n=1}^{\infty} [-n^2 a_n \cos nx - n^2 b_n \sin nx]$, and we apply Parseval's theorem to these functions to find

sim. seen ↓

$$E = -\pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) + \varepsilon \pi \sum_{n=1}^{\infty} n^4 (a_n^2 + b_n^2)$$

hence

$$\pi^{-1} E = \sum_{n=1}^{\infty} n^2 (\varepsilon n^2 - 1) (a_n^2 + b_n^2)$$

2, C

2, D

- (iv) The only term that can be negative is $\varepsilon n^2 - 1$ and since $n \geq 1$ we can see that if $\varepsilon \geq 1$ then all the terms in the infinite series for E are positive.

unseen ↓

2, D

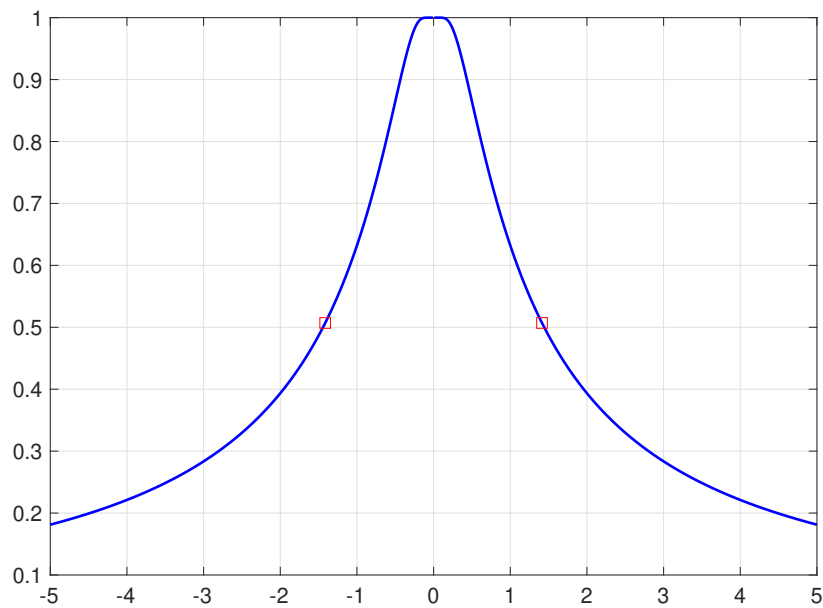


Figure 1: Figure for problem 1(b)(iii). Inflection points denoted by squares.

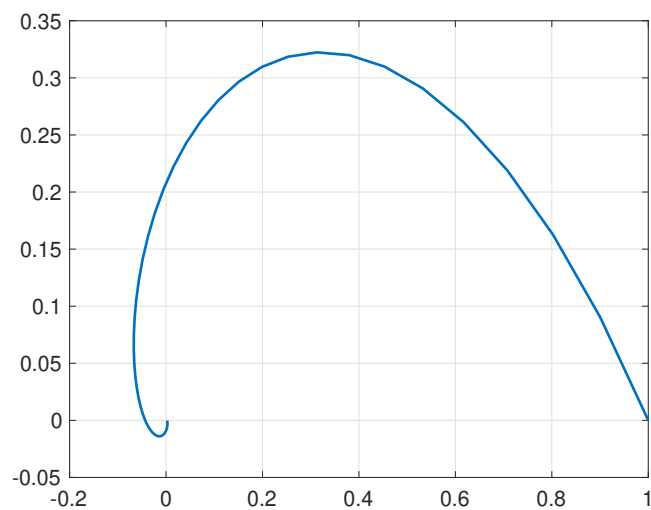


Figure 2: Figure for problem 2(c).

Review of mark distribution:

Total A marks: 24 of 24 marks

Total B marks: 15 of 15 marks

Total C marks: 9 of 9 marks

Total D marks: 12 of 12 marks

Total marks: 60 of 60 marks

Total Mastery marks: 0 of 20 marks

Question
4

Topic Systems of odes + Fourier Transforms

Marks &
seen/unseen

Parts

(a). Let's write the system as: $\frac{d}{dt}y = Ay$, where:
 $y = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $A = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$.

(i). We seek the eigenvalues of A :

$$\begin{vmatrix} -2-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (-2-\lambda)(3-\lambda) - (-4) = 0$$

$$\Rightarrow \lambda^2 - \lambda - 2 = 0 \Rightarrow \underline{\lambda = -1 \text{ or } \lambda = 2}.$$

And their corresponding eigenvectors:

$$\underline{\lambda = -1}: \begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = v_2,$$

so that: $\underline{v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ is an eigenvector.

$$\underline{\lambda = 2}: \begin{pmatrix} -4 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4v_1 = v_2,$$

so that: $\underline{v_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}}$ is an eigenvector.

Thus, we can write the general solution as:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}_6 = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

$c_1, c_2 \in \mathbb{R}$
constants

Seen
similar

$\boxed{A}$
e.vals

$\boxed{A}$
method

$\boxed{A}$
e.vecs

Seen
similar

$\boxed{A}$
result

4(a)(i).4

Setter's initials

STB

Checker's initials

Page number

1

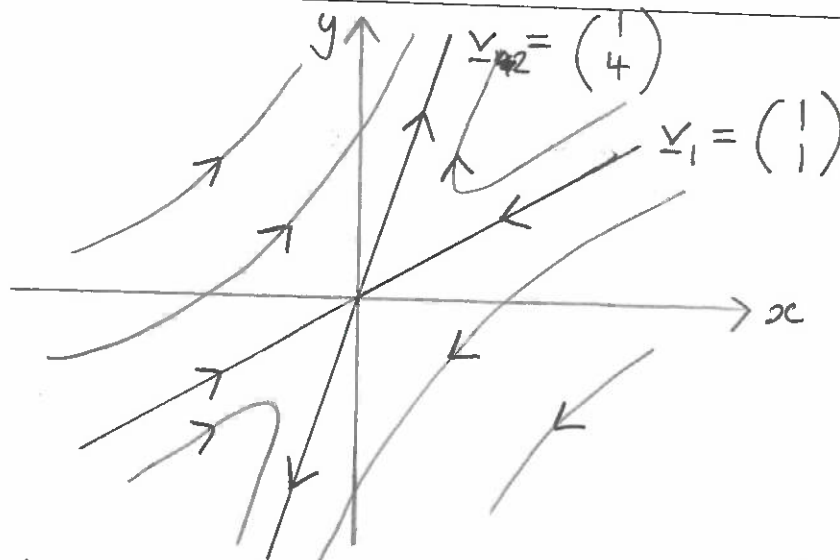
Question
4

Topic Systems of odes + Fourier Transform

Marks &
seen/unseen

Parts

(ii).



We have a saddle point at $(0,0)$; Asymptotically solutions blow up to infinity in the direction of $v_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, unless starting on $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in which case they tend towards the fixed point at $(0,0)$.

(b).

(i).

$$\begin{aligned}\hat{f}_c(\omega) &= \int_0^{\infty} f(x) \cos \omega x \, dx, \text{ by definition.} \\ &= \int_0^1 (1-x) \cos \omega x \, dx.\end{aligned}$$

We now proceed by integration by parts, differentiating the term $(1-x)$ and integrating $\cos \omega x$, we find:

Seen
similar2 A
e-vectors
trajectories1 B
directionsSeen
similar1 BQ4(a)(ii) (4)

definition

2 Buse of $f(x)$
+ range.Seen
similar

Setter's initials

SJB

Checker's initials

Page number

2

	EXAMINATION SOLUTION 23 - 24	Course: Calculus and Applications
Question 4	Topic <i>Systems of odes + Fourier Transforms</i>	Marks & seen/unseen
Parts (b). (i). (continued.)	$\hat{f}_c(\omega) = \left[(1-x) \frac{1}{\omega} \sin \omega x \right]_0^1 + \int_0^1 \frac{1}{\omega} \sin \omega x \, dx$ $= \frac{1}{\omega} \left[-\frac{1}{\omega} \cos \omega x \right]_0^1$ $= -\frac{1}{\omega^2} (\cos \omega - 1)$ $= \frac{1 - \cos \omega}{\omega^2}$ (ii). Since $f(x)$ is an even function, we know that: $\hat{f}(\omega) = 2\hat{f}_c(\omega)$, hence: $\hat{f}(\omega) = \frac{2(1 - \cos \omega)}{\omega^2}$. The energy theorem states that: $\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) ^2 \, d\omega = \int_{-\infty}^{\infty} (f(x))^2 \, dx,$ hence, applying this to $\hat{f}(\omega)$ and $f(x)$: $\frac{1}{2\pi} \int_{-\infty}^{\infty} \left \frac{2(1 - \cos \omega)}{\omega^2} \right ^2 \, d\omega = \int_{-1}^1 (1 - x)^2 \, dx$	$2 \quad \boxed{A}$ seen similar $\boxed{Q4(b)(i)-(4)}$ $1 \quad \boxed{C}$ seen similar $2 \quad \boxed{D}$ unseen use of energy theorem
	Setter's initials SJB	Page number 3

	EXAMINATION SOLUTION 23 - 24	Course: Calculus and Applications
Question 4	Topic Systems of odes + Fourier Transforms	Marks & seen/unseen
Parts (b). (ii). (continued.)	$\Rightarrow \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1 - \cos \omega}{\omega^2} \right)^2 d\omega = 2 \int_0^1 (1-x)^2 dx,$ since $(1- x)^2$ is even. $\Rightarrow \frac{4}{\pi} \int_0^{\infty} \left(\frac{1 - \cos \omega}{\omega^2} \right)^2 d\omega = 2 \left[-\frac{1}{3} (1-x)^3 \right]_0^1 = \frac{2}{3},$ since $\left(\frac{1 - \cos \omega}{\omega^2} \right)^2$ is even. $\Rightarrow \underline{\underline{\int_0^{\infty} \left(\frac{1 - \cos u}{u^2} \right)^2 du = \frac{\pi}{6}}}, \text{ as required.}$ (iii). $\hat{f}(\omega) = \frac{2(1 - \cos \omega)}{\omega^2}$ $= \frac{2}{9} g\left(\frac{\omega}{3}\right).$ The symmetry formula states that: $F\{\hat{f}(x)\} = 2\pi f(-\omega)$, hence: $F\left\{ \frac{2}{9} g\left(\frac{x}{3}\right) \right\} = 2\pi f(-\omega)$ $\Rightarrow F\left\{ g\left(\frac{x}{3}\right) \right\} = 9\pi f(-\omega)$ $= 9\pi f(\omega), \text{ since } f \text{ is even.}$	unseen 1 C even function integrands Q4(b)(ii). (4) 1 C unseen relate $\hat{f}$ to g 2 D -Symmetry formula + manipulation
	Setter's initials SJB	Checker's initials Page number 4

	EXAMINATION SOLUTION 23 - 24		Course: Calculus and Applications
Question 4	Topic Systems of odes + Fourier Transforms		Marks & seen/unseen
Parts (b). (iii). (continued.)	Finally, we use the property $F\{f(ax)\} = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right)$, to conclude: $F\left\{g\left(\frac{x}{3}\right)\right\} = 3\hat{g}(3\omega) = 9\pi f(\omega)$ $\Rightarrow \underline{\hat{g}(\omega) = 3\pi f\left(\frac{\omega}{3}\right)}, \text{ or}$ $\hat{g}(\omega) = \begin{cases} 3\pi(1 - \frac{\omega}{3}), & \omega \leq 3 \\ 0, & \omega > 3. \end{cases}$		D 1 unseen Q4(b)(iii) 4 Q4 total 20
	Setter's initials SJB Checker's initials		Page number 5

Question
5

Topic 1D bifurcations + constant-coefficient ODEs.

Marks &
seen/unseen

Parts

(a). $\frac{dx}{dt} = x^3 - (r-1)x, \quad r \in \mathbb{R}$

(i). Fixed points, x^* , occur when $\frac{dx}{dt} = 0$

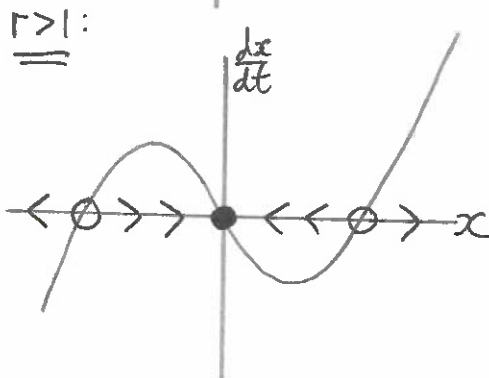
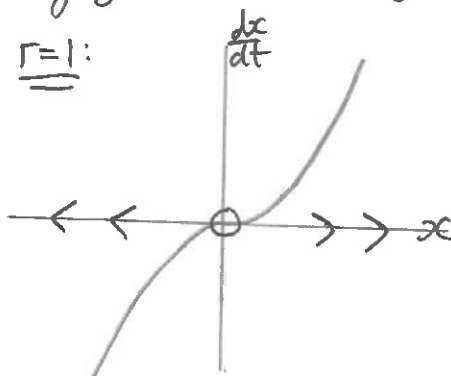
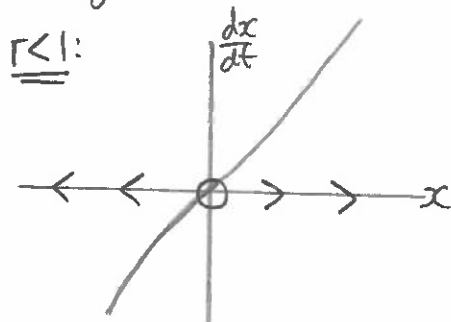
$$\Rightarrow x^{*3} - (r-1)x^* = 0$$

$$\Rightarrow x^* (x^{*2} - (r-1)) = 0$$

$$\Rightarrow \underline{x^* = 0}, \quad \underline{x^* = \pm \sqrt{r-1}}$$

So we see for $r > 1$ three fixed points appear where for $r < 1$ we have only one fixed point.

We graph the phase plane to gauge their stability:



• For $r \leq 1$ there is one fixed point at $x^* = 0$ which is unstable.

• For $r > 1$ there are three fixed points. The one at $x^* = 0$ is stable, while the points at $x^* = \pm \sqrt{r-1}$ are unstable.

A

2

seen
similarseen
similar

B

2

- stability
- ranges of r
- # of fixed points.

(Q5(a)(i) 4)

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SJB

Checker's initials

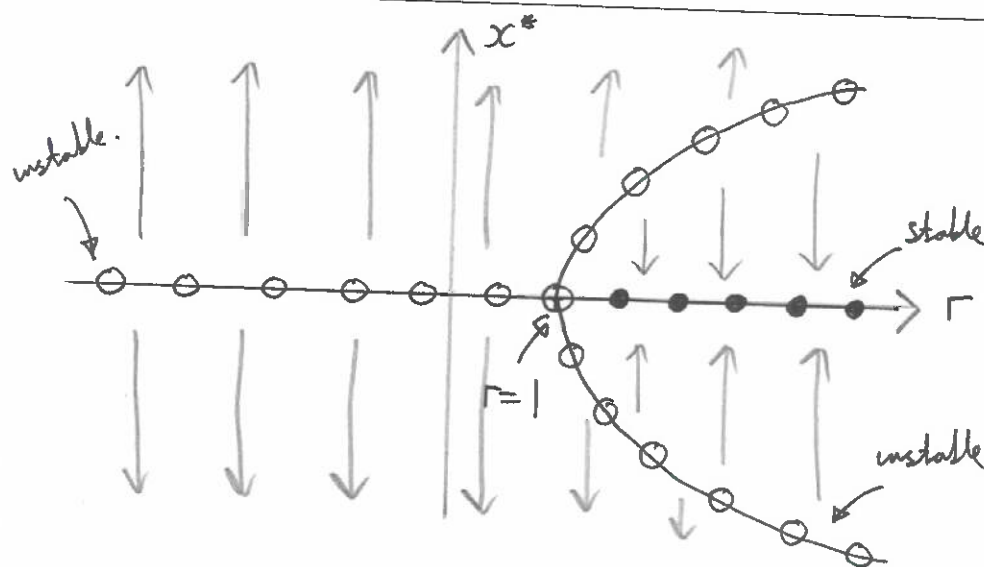
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6

Question
5

Topic 1D bifurcations + constant-coefficient odes.

Parts

(ii).



There is a (subcritical) pitchfork bifurcation at $r = 1$.

Seen
similar

2 [A]

- shape
- stability

1 [B]

 $r = 1$ bif.
point.
+ arrows.

1 [B]

pitchfork.

Q5(a)(ii). (4)

(b).

(i). From lectures we know that the substitution $x = e^z$ results in:

$$x \frac{dy}{dx} = \frac{dy}{dz},$$

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}.$$

We calculate how the third-order term transforms now:

$$x^3 \frac{d^3 y}{dx^3} = x^3 \frac{d}{dx} \left(\frac{1}{x^2} \left[\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right] \right)$$

$$= x^3 \cdot \frac{1}{x^2} \left[\frac{d^3 y}{dx dz^2} - \frac{d}{dx} \frac{dy}{dz} \right] + x^3 \left(-\frac{2}{x^3} \right) \left[\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right]$$

Seen

1 [A]

+ tidy up
at end.

unseen

attempt
at 11/11

1 [C]

Setter's initials

SJB

Checker's initials

Page number

7

EXAMINATION SOLUTION 23 - 24		Course: Calculus and Applications
Question 5	Topic 1D big questions + constant-coefficient odes.	Marks & seen/unseen
Parts (b). (i). (continued.)	$= x \frac{dz}{dx} \left[\frac{d^3 y}{dz^3} - \frac{d^2 y}{dz^2} \right] - 2 \left[\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right]$ $= \frac{d^3 y}{dz^3} - 3 \frac{d^2 y}{dz^2} + 2 \frac{dy}{dz}.$ Hence, substituting $x = e^z$ into the ode gives: $\left(\frac{d^3 y}{dz^3} - 3 \frac{d^2 y}{dz^2} + 2 \frac{dy}{dz} \right) + \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right)$ $= 1 + 8 \cosh z$ $\Rightarrow \frac{d^3 y}{dz^3} - 2 \frac{d^2 y}{dz^2} + \frac{dy}{dz} = 1 + 8 \cosh z,$ as required. (ii) First we find the complementary function. The characteristic equation is: $\lambda^3 - 2\lambda^2 + \lambda = 0$ $\Rightarrow \lambda(\lambda-1)^2 = 0, \text{ so we find:}$ $\underline{y_c(z) = c_1 e^z + c_2 z e^z + c_3}$ solves the homogeneous part. Next we seek a particular integral to solve the non-homogeneous part. The RHS consists of the terms 1, and $\cosh z = \frac{1}{2}e^z + \frac{1}{2}e^{-z}$, and	unseen 2 D correct calculation of $x^3 y'''$ Q5(b)(i). (4) 2 A seen similar
Setter's initials SJB Checker's initials		Page number 8

	EXAMINATION SOLUTION 23 - 24	Course: Calculus and Applications
Question 5	Topic 1D bifurcations + Constant-coefficient odes.	Marks & seen/unseen
Parts (b). (ii). (continued).	both 1 and e^z appear in the complementary solution. Hence we make the ansatz: $y_p(z) = Az + Be^{-z} + Cz^2e^z$ where we have multiplied the 1 and e^z terms by appropriate polynomials. Here A, B and C are to be found. we plug $y_p(z)$ into the ode Differentiating: $y_p'(z) = A - Be^{-z} + 2Cze^z + Cz^2e^z$ $y_p''(z) = Be^{-z} + 2Ce^z + 4Cze^z + Cz^2e^z$ $y_p'''(z) = -Be^{-z} + 6Ce^z + 6Cze^z + Cz^2e^z$ Plugging these into the ode: $\begin{aligned} \text{LHS} &= Cz^2e^z + 6Cze^z + 6Ce^z - Be^{-z} \\ &\quad - 2Cz^2e^z - 8Cze^z - 4Ce^z - 2Be^{-z} \\ &\quad + Cz^2e^z + 2Cze^z - Be^{-z} + A \\ &= 2Ce^z - 4Be^{-z} + A \\ \text{RHS} &= 4e^z + 4e^{-z} + 1 \end{aligned}$ Comparing coefficients of like terms gives: $\underline{A=1, B=-1, C=2}$	} 2 D Seen similar ansatz } 2 B Seen similar } 1 C Compare coefficients
	Setter's initials SJB Checker's initials	Page number 9

	EXAMINATION SOLUTION 23 - 24	Course: Calculus and Applications
Question 5	Topic 1D bifurcations + constant-coefficient odes.	Marks & seen/unseen
Parts (b). (ii). (Continued.)	Leading to: $y_p(z) = \cancel{z^2} z - e^{-z} + 2z^2 e^z$ This has general solution: $y_G(z) = y_c(z) + y_p(z)$, which, written back in terms of the original variable x, is given by: <u>$y_G(x) = C_1 x + C_2 x \log x + C_3 + \log x - \frac{1}{x} + 2x(\log x)^2$</u>	put back into $y(x)$. $\left. \begin{array}{l} C \\ 1 \end{array} \right\}$ seen similar QS(b)(ii). (8) QS total 20
	Setter's initials SJB	Checker's initials Page number 10

	EXAMINATION SOLUTION 23 - 24	Course: Calculus and Applications
Question 6	Topic Exact odes + Stationary points of Surfaces	Marks & seen/unseen
Parts	(a). $\underbrace{(12+5xy)}_M dx + \underbrace{\left(\frac{6x}{y} + 3x^2\right)}_N dy = 0$ (i). We test to see if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$: $\left. \begin{aligned} \frac{\partial M}{\partial y} &= 5x \\ \frac{\partial N}{\partial x} &= \frac{6}{y} + 6x \end{aligned} \right\} \neq$ These expressions are not equal, and hence the ode is <u>not</u> exact. (ii). We multiply through by the integrating factor gives: $\underbrace{(12x^m y^n + 5x^{m+1} y^{n+1})}_M dx + \underbrace{(6x^{m+1} y^{n-1} + 3x^{m+2} y^n)}_N dy = 0.$ $\left. \begin{aligned} \frac{\partial M}{\partial y} &= 12n x^m y^{n-1} + 5(n+1) x^{m+1} y^n \\ \frac{\partial N}{\partial x} &= 6(m+1) x^m y^{n-1} + 3(m+2) x^{m+1} y^n \end{aligned} \right\} =$ For the ode to be exact, we require these expressions to be equal. This leads to the conditions: $12n = 6(m+1), \text{ and}$ $5(n+1) = 3(m+2), \text{ for consistency.}$ Solving these simultaneously one finds: $\underline{\underline{m=3}}$ $\underline{\underline{n=2}}.$	Seen Similar A 2 Q6(a)(i).2 D 3 unseen
	Setter's initials SJB	Page number 11

	EXAMINATION SOLUTION 23 - 24	Course: Calculus and Applications
Question 6	Topic <i>Exact odes + Stationary points of surfaces</i>	Marks & seen/unseen
Parts (a). (ii). (continued.)	Thus the ode: $(12x^3y^2 + 5x^4y^3) dx + (6x^4y + 3x^5y^2) dy = 0,$ is now exact. Hence, seeking an implicit solution $u(x,y)$, we have: $u(x,y) = \int (12x^3y^2 + 5x^4y^3) dx$ $= 3x^4y^2 + x^5y^3 + f(y), \text{ but}$ $6x^4y + 3x^5y^2 = \frac{\partial u}{\partial y} = 6x^4y + 3x^5y^2 + f'(y) \text{ upon}$ differentiating the above. This means that $f'(y) = 0$, and so $f(y) = C$, constant, and we can represent the solution to the ode in the form: $u(x,y) = \underline{\underline{3x^4y^2 + x^5y^3 + C = 0.}}$ (b). $u(x,y) = (2y^2 + 2x - y - 1)(2x - y + 1).$ (i). The zero contour is formed when $u(x_*, y_*) = 0$. $\Rightarrow (2y_*^2 + 2x_* - y_* - 1) = 0, \text{ or } 2x_* - y_* + 1 = 0.$ So the line: $y_* = 2x_* + 1$ and the parabola: $\underline{\underline{x_* = -y_*^2 + \frac{1}{2}y_* + \frac{1}{2}}} \text{ form the zero-contour.}$	A 3 Seen Similar Q6(a)(ii) 6 A 1 Q6(b)(i) 1
	Setter's initials SJB Checker's initials	Q6(b)(i) 1 Page number 12

	EXAMINATION SOLUTION 23 - 24	Course: Calculus and Applications
Question 6	Topic Exact odes + Stationary points of Surfaces	Marks & seen/unseen
Parts	(ii). $\frac{\partial u}{\partial x} = 2(2x-y+1) + 2(2y^2+2x-y-1)$ $= 4y^2 - 4y + 8x$ ① $\frac{\partial u}{\partial y} = (4y-1)(2x-y+1) - (2y^2+2x-y-1)$ $= -6y^2 + 8xy + 6y - 4x$ ② To find stationary points of $u(x,y)$, we seek: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$, i.e. ① = 0 and ② = 0. Plugging ① = 0 for x into ② = 0 gives: $-6y^2 + y(4y - 4y^2) + 6y + 2y^2 - 2y = 0$ $\Rightarrow -4y^3 + 4y = 0$ $\Rightarrow \underline{y = 0, y = \pm 1}$ Plugging these values for y into ① = 0 gives the corresponding values for x, so we find there are <u>three</u> stationary points: <u>$P_1: (0,0), P_2: (0,1), P_3: (-1,-1)$</u> We evaluate the second derivatives of $u(x,y)$ to determine the Hessian matrix: $\frac{\partial^2 u}{\partial x^2} = 8,$	2 A Seen Similar 2 C Seen Similar
	Setter's initials SJB	Checker's initials Page number 13

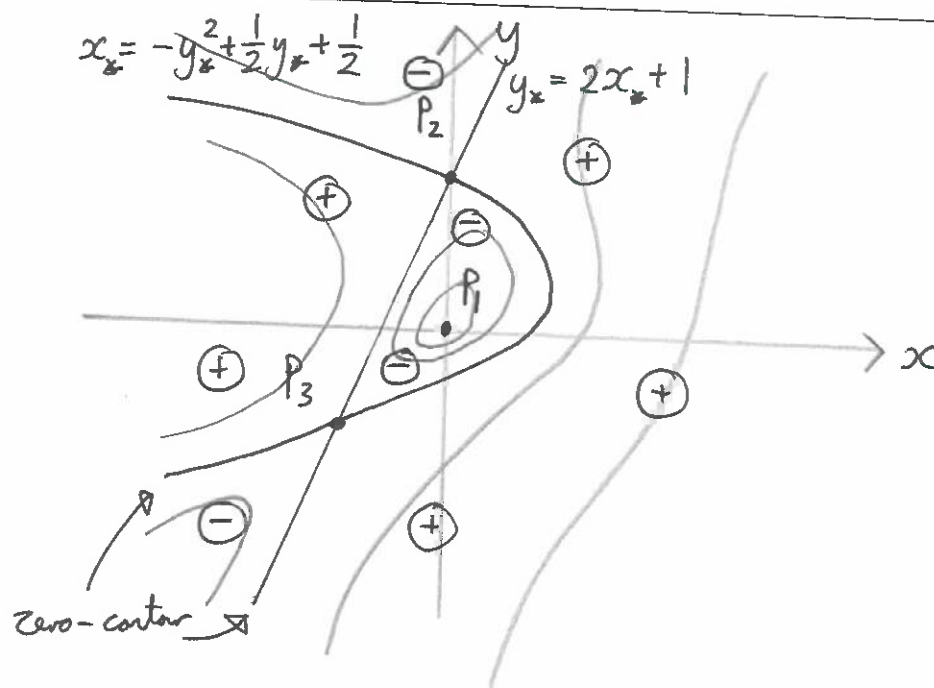
	EXAMINATION SOLUTION 23 - 24	Course: Calculus and Applications
Question 6	Topic <i>Exact odes + Stationary points of Surfaces</i>	Marks & seen/unseen
Parts (b). (ii). (continued.)	$\frac{\partial^2 u}{\partial y^2} = -12y + 8x + 6,$ $\frac{\partial^2 u}{\partial x \partial y} = 8y - 4, \text{ so we find:}$ $H = \begin{pmatrix} 8 & 8y - 4 \\ 8y - 4 & -12y + 8x + 6 \end{pmatrix}.$ At $P_1(0,0)$: $H = \begin{pmatrix} 8 & -4 \\ -4 & 6 \end{pmatrix}$: $\tau = 14, \Delta = 32 > 0$ $\Rightarrow P_1$ is a minimum At $P_2(0,1)$: $H = \begin{pmatrix} 8 & 4 \\ 4 & -6 \end{pmatrix}$: $\tau = 2, \Delta = -64 < 0$ $\Rightarrow P_2$ is a Saddle point. At $P_3(-1,-1)$: $H = \begin{pmatrix} 8 & -12 \\ -12 & 10 \end{pmatrix}$: $\tau = 18, \Delta = -64 < 0$ $\Rightarrow P_3$ is a Saddle point.	A 1 -Hessian Seen Similar B 2 classification Q6(b)(ii). 7
	Setter's initials SJB	Checker's initials
		Page number 14

Question
6

Topic Exact ODEs + Stationary points of Surfaces

Parts

(iii).



In part (ii) we saw P_1 was a minimum, our plot shows that $u(P_1) < 0$ and P_1 lies in a region of negative u which is consistent with P_1 being a minimum.

We saw P_2 and P_3 were saddles in part (ii). Indeed we see in both cases the value of u becomes negative in certain directions while it becomes positive in other directions. This is consistent with each of these being saddle points.

Seen
similar

B

3

- zero contour
- points.
- signs.
- shape.

Seen
similar

C

1

reasonable
explanation

Q6(b)(iii).4

Q6 total
20

Setter's initials

SJB

Checker's initials

Page number

15

Question Marker's comment

- 1 The main issue where a lot of people followed the wrong path, has to do with taking the derivative of $\exp(-1/|x|)$ - or in general $f(|x|)$. Many students treated $|x|$ as the independent variable and differentiated with respect to $|x|$. This is wrong - it needs to be done separately for $x > 0$ and $x < 0$. The example $y = |x|$ illustrates this - the derivative is not 1 for all x .
- 2 (a) Almost everybody differentiated $\log(1/\cos(x))$ correctly. The Riemann sum was done far less well. Many students recognised that the integral of $\tan(x)$ from 0 to $\pi/4$ was relevant but failed to realise that $1/n$ was not the correct width for the partition of $[0, \pi/4]$ so failed to obtain the correct sum. (b) (i) While almost everybody knew how to obtain the Taylor series of $\tan(x)$, there were often various small errors in differentiation. A surprising minority were unable to use the chain rule correctly here. A mark was deducted for such errors even if they happened to lead to the correct answer. Almost nobody attempted to divide the Taylor series of $\sin(x)$ and $\cos(x)$ which is a useful technique to know. (ii) This was much less well done. A rather significant minority could not recall the definition of centre of mass. The non-uniform density also caused confusion. Since the integral J was stated in the question, no marks were awarded for answers such as $\bar{x} = \rho_0 J$ or $\bar{x} = \rho_0 J / (\pi/4)$. Some understanding of mass $\times \bar{x} = \text{moment}$ was required to obtain any marks. Those who understood the concept were generally able to obtain 4/4, though there were occasional errors in integrating $\tan(x)$ (despite the anti-derivative being explicitly given in part(a)). (iii) This was generally well done. Some marks were awarded for a correct approach to J to even if (ii) was not done. Unfortunately there was an unannounced typo in the question which affected the final result (namely the 90 should have been an 80 in the given answer). However, this did not seem to have affected the students significantly and anyone who showed enough correct work was awarded full marks. (c) This was generally well done, though full marks were rare. Most students were able to sketch the curve correctly. The discussion of the behaviour as t tends to infinity was less well done, but a mark was given for any indication of spiralling into the origin. Those who drew this in the sketch often failed to mark the point $t = 2\pi$ as required by the question (and those who only sketched up to $t = 2\pi$ often failed to provide a sufficient discussion). The length calculation was well done, with the occasional minor slip. A minority did not recall the formula correctly or attempted to use the non-parametric form.
- 3 Question 3 centres around Fourier series and the key point is to do the integral properly. For (a) part, it is good that most of the students know how to get the coefficients of the Fourier series for a periodic odd function. But when considering what values the Fourier series converges to, it is easier to look at the function instead of calculating it from the Fourier series. It should be checked carefully that whether the function is continuous at this point as well. The Fourier series converges to the average value of the function at points of discontinuity. For (b) part, some of the students know the general but not exact Fourier series representation of Parseval's theorem. But they can use the integrating by parts to finish the proof (ii). One thing should be pointed in the process of the proof is how to cancel some of the terms. It is important to understand the given conditions: what the periodicity and zero spatial mean are. The zero spatial mean is $a_0 = 0$. For (iv), it should be noted that when $\epsilon \{n^2\}^{-1} \geq 0$ for $n = 1$, then for other terms are larger than 0.

Question Marker's comment

- 4 Common mistakes in this question:
 - 4ai) Was well done by most of the course, no comments about this part.
 - 4aii) In general, it was well done. Common mistakes in the phase portrait are forgetting to label the lines defined by the eigenvectors and drawing the arrows in the wrong direction. Some people forget to include a description of the asymptotic behaviour. A fully correct description should include: what happens if $x(0)=y(0)$ (the trajectory goes to the origin along the line $y=x$), what happens if $y(0) = 4x(0)$ (the trajectory diverges to infinity along the line $y=4x$) and otherwise the trajectories also go to (plus/minus) infinity converging asymptotically to the line $y=4x$.
 - 4bi) In this part there were two very common mistakes: (1) computing the Fourier Transform and not writing the cosine Fourier transform, as asked in the exercise, (2) computing the cosine Fourier transform as an integral from $-\infty$ to $+\infty$, while its definition is with an integral from 0 to $+\infty$.
 - 4bii) Most people got the strategy right (using the Energy Theorem + the relation between the FT and the cosine FT computed in 4bi). Many people loose marks in computations, by not keeping track of the correct factors. Since the answer it is known beforehand, many students attempt to fix this changing factors from one line to another without justifying.
 - 4biii) Many people get the relation between $g(x)$ and the FT of $f(x)$, although there are common mistakes in getting the multiplicative factors right (many people forget the 2 to relate to the cosine FT computed in 4bi or don't use the correct arguments to relate the functions, for instance). Many people use the Symmetry Formula wrong, without checking if the arguments in the functions make sense; in other words, they forget to use the relation $F\{f(ax)\} = 1/a \wedge f(w/a)$.
- 5 Overall, this question was answered very well by students. The most common mistakes involved incorrectly calculating the third order derivative in (b)(i) (or omitting to show any calculation, despite this being a 'show that' question), or students missing that $\cosh(z) = (e^z + e^{-z})/2$ in part (b)(ii), which was important to form the correct ansatz for the particular integral. Another common error appeared in (a)(ii), where many scripts omitted defining the type of bifurcation.
- 6 Question 6 was answered to varying degrees of success, although this was the final question. 6(a) had very few issues, where the only common error seemed to be in solving for m and n in the integrating factor. However people that found incorrect values still seemed to understand the method in finding the general solution, and as a result scored respectfully here. On the other hand, 6(b) had a plethora of issues. In b(i), a number of students seemed convinced that they also had to find points of intersection in order to score a mark, though this did not result in any penalisation if the zero contours were correctly identified. b(ii) and b(iii) were by far the most polarising parts, with students rarely achieving full marks due to a number of errors seen in most papers. Surprisingly for b(iii), students that drew a high scoring contour plot seemed reluctant to write any form of argument for the consistency of their classifications of each fixed point, suggesting some sort of lack of understanding of what the contour plot actually meant.