

MATH40005

Probability and Statistics

Revised Mock Examination 1

Time allowed	2 hours
Questions	4
Total marks	80
Format	Closed book; justify all answers

- Answer all four questions.
- Start each question on a new page in an actual timed attempt.
- State assumptions and named results used.
- Statistical tables or stated critical values may be used where supplied.

Question 1

1(a). A box contains cards labelled $1, 2, \dots, 8$. One card is drawn, replaced, and a second card is drawn. Briefly describe a probability space $(\Omega, \mathcal{F}, P)$ when all cards are equally likely. (3 marks)

1(b)(i). A password has 10 characters: 5 letters chosen from 52 letters with repetition allowed, 3 digits chosen from 10 digits with repetition allowed, and 2 distinct symbols chosen from 5 symbols. Count all possible passwords. (5 marks)

1(b)(ii). Assuming all pairs of symbol positions are equally likely, find the probability that the two symbols are adjacent. (4 marks)

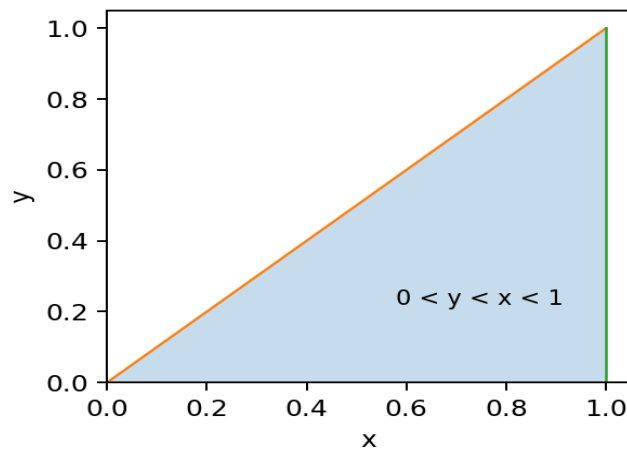
1(c). Let r be an integer and $0 < p < 1$. A random variable has pmf $p_X(x) = p(1-p)^{(x-r)}$ for $x = r, r+1, \dots$ and zero otherwise. Verify that it is a valid pmf and find its CDF for every real x . (5 marks)

1(d). There are n independent components, each failing on a specified day with probability $1/d$. Let Y_i indicate failure of component i and $B = \sum Y_i$. Identify the distributions of Y_i and B , including parameters. (3 marks)

Total: 20 marks

Question 2

2(a). For $0 < y < x < 1$ let $f_{XY}(x,y)=c(x+y)$, and let it be zero otherwise. Find c . (3 marks)



2(b). Find the marginal densities of X and Y . (4 marks)

2(c). Find the conditional density of Y given $X=x$, and state where it is defined. (3 marks)

2(d). Are X and Y independent? Give a decisive reason. (2 marks)

2(e). Compute $\text{Cov}(X,Y)$. (5 marks)

2(f). Let $R=Y/X$. Find the CDF of R . (3 marks)

Total: 20 marks

Question 3

3(a). Let X have finite first and second moments. State the minimizer of $E[(X-a)^2]$, the minimum value, and a minimizer of $E|X-b|$. (3 marks)

3(b). Prove the bias-variance decomposition for an estimator $\hat{\theta}$ of a fixed parameter θ . (3 marks)

3(c). Fifteen independent observations are assumed normal with unknown mean and variance. The sample mean is 74.2 and sample variance is 6.25. Compute a 95% confidence interval for the mean. Use $t_{(14,0.975)}=2.145$. (4 marks)

3(d). Using the same data, test $H_0: \mu=75$ against $H_1: \mu \neq 75$. Compute a two-sided p-value and state the conclusion at level 0.05. (5 marks)

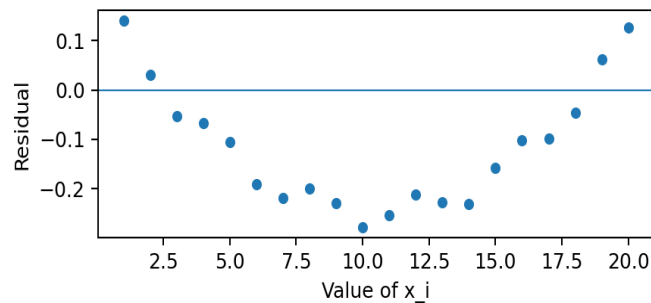
3(e). Define a p-value, explain how it is interpreted, and state when it is $\text{Uniform}(0,1)$. (5 marks)

Total: 20 marks

Question 4

4(a). Suppose $X \sim N(5, 9)$, $Y \sim N(4, 4)$, and $\text{Corr}(X, Y) = -1/3$. Let $Z = X - 2Y$. Compute $\text{Corr}(X, Z)$. (4 marks)

4(b). Let X and Y have equal finite variances. Prove that $X + Y$ and $X - Y$ are uncorrelated. (5 marks)



4(c). A simple linear regression has $R^2 = 0.94$ and the residual plot shown. Does the model fit well? Suggest a specific improvement. (4 marks)

4(d). Independent observations are modeled as $\text{Uniform}[\theta, \theta + 2]$. Assuming $\max(x_i) - \min(x_i) \leq 2$, find the full set of maximum likelihood estimates of θ . (5 marks)

4(e). For original data $\{12, 18, 21, 27, 30\}$, decide whether (i) $\{18, 18, 30, 12, 27\}$ and (ii) $\{12, 18, 21, 27\}$ are bootstrap samples. Justify. (2 marks)

Total: 20 marks