

MATH40005

Probability and Statistics

Revised Mock Examination 3

Time allowed	2 hours
Questions	4
Total marks	80
Format	Closed book; justify all answers

- Answer all four questions.
- Start each question on a new page in an actual timed attempt.
- State assumptions and named results used.
- Statistical tables or stated critical values may be used where supplied.

Question 1

1(a). Assume each birthday week is independent and uniformly distributed over 52 weeks. For a class of 20 pupils, find the probability that at least two share a birthday week. (5 marks)

1(b). Assume birthdays are independent and uniform over 365 days. For $3 \leq n \leq 300$, find the probability that exactly one pair shares a birthday and all other birthdays are distinct. (5 marks)

1(c). A sick child has flu with probability 0.95 and measles with probability 0.05. A rash occurs with probability 0.01 under flu and 0.90 under measles. Given a rash, find the probability of measles. (5 marks)

1(d). Let X and Y be discrete random variables and A an event. Define $Z=X$ on A and $Z=Y$ on A^c . Prove that Z is a discrete random variable. (5 marks)

Total: 20 marks

Question 2

2(a). Let $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$, but do not assume independence. Must $X+Y$ be Poisson? Give a proof or counterexample. (4 marks)

2(b). Let $N \sim \text{Poi}(\lambda)$, independent of i.i.d. $X_i \sim \text{Uniform}(0,1)$. For $0 < x < 1$, find $P(\min\{X_1, \dots, X_{N+1}\} > x)$. (6 marks)

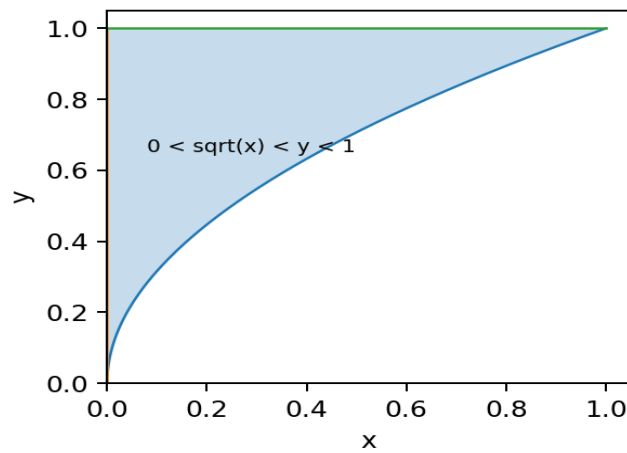
2(c). Let X, Y be independent standard normal variables. Find $P(X > 0, Y > -X)$. (5 marks)

2(d). Calls arrive as $N \sim \text{Poi}(\lambda)$. Independently, each call is labelled urgent with probability p and routine otherwise. Let U and R be the counts. Show that $U \sim \text{Poi}(\lambda p)$, $R \sim \text{Poi}(\lambda(1-p))$, and that U and R are independent. (5 marks)

Total: 20 marks

Question 3

3(a). For $f_{XY}(x,y)=c(x+y)$ on $0 < \sqrt{x} < y < 1$ and zero otherwise, find c , both marginal densities, and decide whether X and Y are independent. (8 marks)



3(b). For $0 < x < 1$, find the conditional CDF $P(Y \leq y | X=x)$, including all cases in y . (5 marks)

3(c). Let $X \sim \text{Uniform}(1,3)$ and $W=X^4$. Find the CDF and density of W , and compute $E[W]$. (7 marks)

Total: 20 marks

Question 4

4(a). Let $N \sim \text{Poi}(\lambda)$, independent of i.i.d. Y_i with mean m and variance v . For $S = \sum_{i=1}^N Y_i$ (zero when $N=0$), derive $E[S]$ and $\text{Var}(S)$. (8 marks)

4(b). Twelve glasses contain six cola and six other drinks. A person selects six as cola and gets five correct. Under random guessing, perform a one-sided exact test of ability at level 0.05. (5 marks)

4(c). A normal population has known standard deviation 6. A random sample of $n=36$ has $\bar{x}=102$. Construct a 95% confidence interval for μ . (4 marks)

4(d). Explain the frequentist meaning of a 95% confidence interval after it has been computed. (3 marks)

Total: 20 marks