

MATH40005

Probability and Statistics

Revised Mock Examination 4

Time allowed	2 hours
Questions	4
Total marks	80
Format	Closed book; justify all answers

- Answer all four questions.
- Start each question on a new page in an actual timed attempt.
- State assumptions and named results used.
- Statistical tables or stated critical values may be used where supplied.

Question 1

1(a). Define a sigma-algebra $\mathcal{F}$ on a sample space Ω . (3 marks)

1(b). Define a probability measure P on $(\Omega, \mathcal{F})$. (3 marks)

1(c). How many ordered n -tuples of positive integers sum to k ? State all cases. (4 marks)

1(d). Prove algebraically that $n 4^{n-1} = \sum_{k=0}^n C(n, k) 3^k 1^{n-k}$. (4 marks)

1(e). Give a story proof of the identity in part (d). (6 marks)

Total: 20 marks

Question 2

2(a). A discrete variable has $P(X=1)=1/2$, $P(X=3)=1/3$, $P(X=5)=1/6$. Write its CDF for every real x . (4 marks)

2(b). Let $X \sim \text{Geom}(p_1)$ and $Y \sim \text{Geom}(p_2)$ on $\{1, 2, \dots\}$, independently. Find the distribution of $\min(X, Y)$. (6 marks)

2(c). A fair coin is tossed until the pattern HT first appears. Find the expected number of tosses. (6 marks)

2(d). Prove the law of total expectation for a discrete X and a countable partition $\{B_i\}$ with $P(B_i) > 0$, assuming absolute convergence. (4 marks)

Total: 20 marks

Question 3

3(a). Let X_i be i.i.d. with mean θ and variance σ^2 . As an estimator of θ define $T = (n+1)^{-1} \sum X_i$. Find its bias, variance and MSE. (6 marks)

3(b). Prove Markov's inequality for a non-negative random variable X . (4 marks)

3(c). An unknown Bernoulli proportion p is estimated by the sample proportion. Using Chebyshev, find n sufficient for $|\hat{p} - p| < 0.02$ with confidence at least 0.95 uniformly in p . (5 marks)

3(d). One hundred hypotheses are tested with family-wise level 0.01. Under Bonferroni, which p -values among 0.03, 0.009, 0.0004, 0.00002, 0.000005 are significant? (5 marks)

Total: 20 marks

Question 4

4(a). For data (x_i, y_i) , derive the least-squares estimates in $y_i = \beta_0 + \beta_1 x_i + e_i$. (6 marks)

4(b). For i.i.d. observations from $\text{Uniform}[0, \theta]$, $\theta > 0$, find the MLE of θ . (5 marks)

4(c). Let $X_i | \theta$ be independent $\text{Exponential}(\text{rate } \theta)$ and $\theta \sim \text{Gamma}(a, b)$ in rate parameterization. Derive the posterior. (4 marks)

4(d). Suppose $E[X]=2$, $E[Y]=3$, $E[XY]=4$ and $2 \leq Y \leq 5$. Obtain a non-trivial lower bound on $\text{sd}(X)$. (5 marks)

Total: 20 marks