

MATH40005

Probability and Statistics

Revised Mock Examination 6

Time allowed	2 hours
Questions	4
Total marks	80
Format	Closed book; justify all answers

- Answer all four questions.
- Start each question on a new page in an actual timed attempt.
- State assumptions and named results used.
- Statistical tables or stated critical values may be used where supplied.

Question 1

1(a). Prove by double counting that $n 5^{n-1} = \sum_{k=0}^n C(n,k) 4^k 1^{n-k}$. Give both an algebraic and a story interpretation. (6 marks)

1(b). A test has sensitivity 0.95 and false-positive rate 0.05. Compute $P(\text{disease}|\text{positive})$ when prevalence is (i) 0.01 and (ii) 0.10, and explain the change. (6 marks)

1(c). Assume independent uniform birthdays over 365 days. Find the probability that among 12 people exactly one pair shares a birthday and all others are distinct. (5 marks)

1(d). A fair four-sector spinner is spun and then a fair coin is tossed. Describe a probability space. (3 marks)

Total: 20 marks

Question 2

2(a). For $f_{XY}(x,y)=c x(y+1)$ on $0<x<y<1$, find c , $f_X(x)$, and $E[Y|X=x]$, stating the domain. (8 marks)

2(b). Let $N \sim \text{Poi}(\lambda)$, independent of i.i.d. Y_i with mean m and variance v . Derive the mean and variance of $S = \sum_{i=1}^N Y_i$. (7 marks)

2(c). Let $M = \max(X_1, \dots, X_n)$ for i.i.d. $\text{Uniform}(0,1)$, and define $T = -\log M$. Identify the distribution of T . (5 marks)

Total: 20 marks

Question 3

3(a). Independent $X_i \sim N(\mu_i, \sigma^2)$ may have unequal means. Derive $E[S^2]$ for the usual sample variance. (6 marks)

3(b). Football and basketball heights are independent normal samples with equal unknown variance. Data: $n=16$, $\bar{x}=176.5$, $s_x^2=4$; $m=9$, $\bar{y}=181$, $s_y^2=4$. Test $H_0: \mu_1 + \mu_2 = 360$ against a two-sided alternative at 0.05, giving a p-value. (7 marks)

3(c). For $n=12$ normal observations with $s^2=3$, compute a 99% confidence interval for σ^2 . Use $\chi^2_{11,0.005}=2.603$ and $\chi^2_{11,0.995}=26.757$. (4 marks)

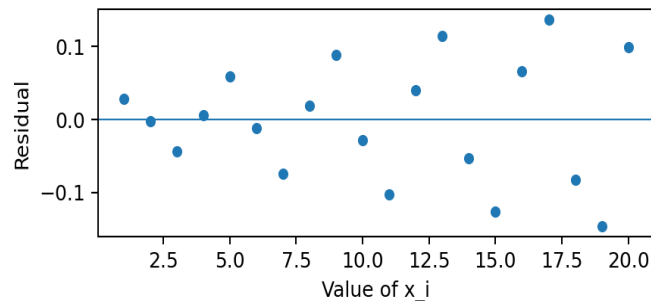
3(d). Define a p-value and prove that an exact upper-tail p-value is $\text{Uniform}(0,1)$ under H_0 when the test statistic has a continuous null CDF F_0 . (3 marks)

Total: 20 marks

Question 4

4(a). For the no-intercept model $y_i = \beta x_i^2 + e_i$, derive the least-squares estimate of β . (5 marks)

4(b). The residual plot shown comes from a fitted straight-line model. Diagnose the problem and give a remedy. (3 marks)



4(c). Independent observations are $\text{Uniform}[\theta, \theta+1]$. Find the full set of MLEs for θ . (5 marks)

4(d). Let $X_i | \theta$ be independent geometric variables with $P(X_i = x) = \theta(1-\theta)^{x-1}$, and $\theta \sim \text{Beta}(a, b)$. Derive the posterior. (4 marks)

4(e). Two unrelated increasing time series have sample correlation 0.97. State what can and cannot be concluded. (3 marks)

Total: 20 marks