

MATH40005

Probability and Statistics

Revised Mock Examination 1 - Solutions and Marking Guide

Time allowed	2 hours
Questions	4
Total marks	80
Format	Closed book; justify all answers
Difficulty calibration	A 32, B 20, C 12, D 16
Paper role	2026 structural reinforcement: one deliberately close paper, but with joint-density and position-counting variants rather than a line-by-line clone.

- Answer all four questions.
- Start each question on a new page in an actual timed attempt.
- State assumptions and named results used.
- Statistical tables or stated critical values may be used where supplied.

Question 1

1(a) (3 marks)

Recognition signal	A finite experiment with equally likely ordered outcomes asks for all three components of a probability space.
First key step	Write the ordered sample space first; replacement means all 8^2 pairs are possible.
Historical basis	2025 Q1(a)(i); 2026 Q1(a)
Ability/method	finite probability-space construction; K01

Course-method solution. Take $\Omega = \{1, \dots, 8\}^2$, $F = \mathcal{P}(\Omega)$, and $P(\{(i, j)\}) = 1/64$. Hence $P(A) = |A|/64$ for every A subset Ω .

Marking points. 1 mark: ordered sample space; 1 mark: event space; 1 mark: probability measure

Common errors. Using unordered pairs; omitting F or P .

Final self-check. Check $|\Omega| = 64$ and the singleton probabilities sum to 1.

Difficulty allocation: A:3; estimated normal time 4 min.

1(b)(i) (5 marks)

Recognition signal	Fixed counts of several character types with repetitions and distinct symbols.
First key step	Separate position choices from character choices.
Historical basis	2025 Q2(a)(i)
Ability/method	multistage multiplication and repeated-element counting; K05

Course-method solution. Choose the two symbol positions, then five of the remaining positions for letters. Order the two distinct symbols. The count is $C(10, 2) C(8, 5) (5 \cdot 4) 52^5 10^3$.

Marking points. 2 marks: choose type positions; 1 mark: order the distinct symbols; 2 marks: character choices and product

Common errors. Using $C(5, 2)$ without ordering the two symbols; forgetting repetitions.

Final self-check. The multinomial form $10!/(5!3!2!)$ times $52^5 10^3$ times $C(5, 2)2!$ must agree.

Difficulty allocation: A:2, B:3; estimated normal time 7 min.

1(b)(ii) (4 marks)

Recognition signal	The random feature depends only on positions, not symbol identities.
First key step	Count adjacent unordered pairs among all unordered position pairs.
Historical basis	2025 Q2(a)(ii)
Ability/method	position-pattern probability; K05

Course-method solution. There are $C(10, 2)$ possible pairs of symbol positions and 9 adjacent pairs: $(1, 2), \dots, (9, 10)$. Therefore the probability is $9/C(10, 2) = 1/5$.

Marking points. 1 mark: denominator; 2 marks: adjacent-pair count; 1 mark: probability

Common errors. Dividing by $10!$ or by the total number of passwords.

Final self-check. The answer must lie between 0 and 1 and should be close to $2/10$.

Difficulty allocation: B:1, C:1, D:2; estimated normal time 6 min.

1(c) (5 marks)

Recognition signal	A shifted lattice distribution requires a floor function in the all-real CDF.
First key step	Re-index with $k = x - r$.
Historical basis	2026 Q1(c); 2020 Q2(c)(i)
Ability/method	support-aware finite/infinite geometric sums; K11

Course-method solution. Non-negativity is immediate. The sum is $p \sum_{k=0}^{\infty} (1-p)^k = 1$. For $x < r$, $F_X(x) = 0$. For $x \geq r$, $F_X(x) = \sum_{k=0}^{\lfloor x \rfloor - r} p(1-p)^k = 1 - (1-p)^{\lfloor x \rfloor - r + 1}$.

Marking points. 1 mark: non-negativity; 1 mark: total mass; 2 marks: finite geometric sum; 1 mark: all-real piecewise CDF

Common errors. Giving the CDF only at integer x ; missing $x < r$.

Final self-check. At $x=r$ the CDF should equal p , and it should tend to 1.

Difficulty allocation: A:2, C:3; estimated normal time 7 min.

1(d) (3 marks)

Recognition signal	A count of independent yes/no events is a sum of indicators.
First key step	Define success as failure on the specified day.
Historical basis	2026 Q2(c)(i)
Ability/method	indicator-to-binomial modelling; K09

Course-method solution. $Y_i \sim \text{Bernoulli}(1/d)$. Since the indicators are independent, $B \sim \text{Binomial}(n, 1/d)$.

Marking points. 1 mark: Bernoulli model; 1 mark: binomial model; 1 mark: parameters and independence

Common errors. Calling B Poisson without an approximation statement.

Final self-check. $E[B] = n/d$ is consistent with the binomial model.

Difficulty allocation: A:2, B:1; estimated normal time 3 min.

Total: 20 marks

Question 2

2(a) (3 marks)

Recognition signal	A joint density with an unknown constant on a triangular support.
First key step	Sketch/read the support and normalize.
Historical basis	2025 Q2(b)(i); 2023 Q3(b)(i)
Ability/method	joint-density normalization on non-rectangular support; K17,K22

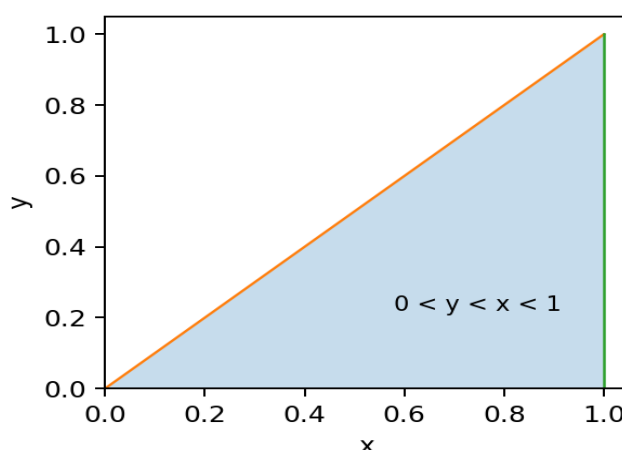
Course-method solution. Integrate over $0 < x < 1$ and $0 < y < x$: $\int_0^1 \int_0^x c(x+y) dy dx = c/2$. Hence $c=2$.

Marking points. 1 mark: correct region/bounds; 1 mark: integral; 1 mark: $c=2$ and non-negativity

Common errors. Integrating over the unit square.

Final self-check. Substitution $c=2$ must make the integral 1.

Difficulty allocation: A:2, B:1; estimated normal time 5 min.



2(b) (4 marks)

Recognition signal	Marginals require integrating along the support slice determined by the other variable.
First key step	For fixed x , y runs 0 to x ; for fixed y , x runs y to 1.
Historical basis	2025 Q2(b)(ii); 2023 Q3(b)(ii)
Ability/method	marginalization on triangular support; K18

Course-method solution. For $0 < x < 1$, $f_X(x) = \int_0^x 2(x+y) dy = 3x^2$. For $0 < y < 1$, $f_Y(y) = \int_y^1 2(x+y) dx = 1 + 2y - 3y^2$. Both are zero otherwise.

Marking points. 2 marks: f_X with support; 2 marks: f_Y with support

Common errors. Reversing the bounds or omitting supports.

Final self-check. Each marginal integrates to 1.

Difficulty allocation: A:2, B:2; estimated normal time 6 min.

2(c) (3 marks)

Recognition signal	The phrase “given $X=x$ ” in a continuous model signals a density ratio.
First key step	Compute f_X first, then divide.
Historical basis	2025 Q2(b)(iii); 2022 Q3(c)(iii)
Ability/method	conditional-density ratio and support; K18

Course-method solution. For $0 < x < 1$ and $0 < y < x$, $f_{Y|X=x}(y|x) = f_{XY}(x,y)/f_X(x) = 2(x+y)/(3x^2)$. It is defined for x with $f_X(x) > 0$, namely $0 < x < 1$.

Marking points. 1 mark: ratio formula; 1 mark: expression; 1 mark: support/domain

Common errors. Treating $P(X=x) > 0$; failing to state $0 < y < x$.

Final self-check. Integrating the conditional density from 0 to x gives 1.

Difficulty allocation: A:1, B:2; estimated normal time 5 min.

2(d) (2 marks)

Recognition signal	Independence must hold on the full support, not only where the joint density is positive.
First key step	Use support or check factorization.
Historical basis	2025 Q2(b)(iv); 2023 Q3(b)(iii)
Ability/method	support/factorization test for independence; K19

Course-method solution. No. The support is triangular, not a Cartesian product: for example $f_{XY}(1/4, 1/2) = 0$ while both marginals are positive. Equivalently, the conditional density depends on x .

Marking points. 1 mark: correct conclusion; 1 mark: decisive support/factorization argument

Common errors. Saying “not independent because x and y appear together” without proof.

Final self-check. Find one point where $f_X f_Y > 0$ but $f_{XY} = 0$.

Difficulty allocation: A:2; estimated normal time 3 min.

2(e) (5 marks)

Recognition signal	Covariance from a joint density requires three moments.
First key step	Write $E[XY] - E[X]E[Y]$ and compute each integral on the same support.
Historical basis	2024 Q2(b)(vi); 2022 Q2(b)
Ability/method	joint LOTUS moments and covariance; K25, K27

Course-method solution. Using LOTUS over $0 < y < x < 1$: $E[X] = 3/4$, $E[Y] = 5/12$, and $E[XY] = 1/3$. Hence $\text{Cov}(X, Y) = 1/3 - (3/4)(5/12) = 1/48$.

Marking points. 1 mark: $E[X]$; 1 mark: $E[Y]$; 2 marks: $E[XY]$; 1 mark: covariance

Common errors. Using marginal means but forgetting $E[XY]$; assuming independence.

Final self-check. The positive support ordering $Y < X$ makes a small positive covariance plausible.

Difficulty allocation: B:1, C:1, D:3; estimated normal time 8 min.

2(f) (3 marks)

Recognition signal	A ratio transformation is most directly handled by the event $Y \leq rX$.
First key step	Determine the range of R before integrating.
Historical basis	2021 Q3(c)(iii); 2023 Q3(b)(iv)
Ability/method	CDF method for a ratio on triangular support; K20, K22

Course-method solution. R lies in $(0, 1)$. For $0 < r < 1$, $F_R(r) = P(Y \leq rX) = \int_0^1 \int_0^{rx} 2(x+y) dy dx = (2r + r^2)/3$. Thus $F_R(r) = 0$ for $r \leq 0$ and 1 for $r \geq 1$.

Marking points. 1 mark: support/cases; 1 mark: correct integration region; 1 mark: evaluated CDF

Common errors. Using $y \leq r$ without multiplying by x ; missing endpoint cases.

Final self-check. At $r=0$ the CDF is 0 and at $r=1$ it is 1.

Difficulty allocation: D:3; estimated normal time 7 min.

Total: 20 marks

Question 3

3(a) (3 marks)

Recognition signal	This asks for the standard loss-minimization characterizations.
First key step	Recall the completed-square identity and the median property.
Historical basis	2024 Q4(a); 2026 Q3(a)
Ability/method	mean/median loss minimizers; K29

Course-method solution. The squared-loss minimizer is $a=E[X]$, and the minimum is $\text{Var}(X)$. Any median of X minimizes absolute loss.

Marking points. 1 mark each for mean, variance, median

Common errors. Confusing the two minimizers.

Final self-check. Squared loss has units squared and its minimum is a variance.

Difficulty allocation: A:3; estimated normal time 3 min.

3(b) (3 marks)

Recognition signal	MSE is the second moment of the estimation error.
First key step	Set $W=\theta_{\text{hat}}-\theta$.
Historical basis	2023 Q4(a); 2026 Q3(b)
Ability/method	bias-variance identity; K31

Course-method solution. Apply $E[W^2]=\text{Var}(W)+(E[W])^2$ to $W=\theta_{\text{hat}}-\theta$. Since $\text{Var}(\theta_{\text{hat}}-\theta)=\text{Var}(\theta_{\text{hat}})$ and $E[\theta_{\text{hat}}-\theta]=\text{Bias}_{\theta}(\theta_{\text{hat}})$, $\text{MSE}=\text{Var}(\theta_{\text{hat}})+\text{Bias}^2$.

Marking points. 1 mark: identity; 1 mark: variance step; 1 mark: bias step

Common errors. Expanding but dropping the cross term without explanation.

Final self-check. Both terms on the right are non-negative.

Difficulty allocation: A:2, B:1; estimated normal time 4 min.

3(c) (4 marks)

Recognition signal	Normal sample, unknown variance, small n: use Student t.
First key step	Take the square root of the sample variance before forming the standard error.
Historical basis	2020 Q6(a)(ii); 2024 Q4(c)(i)
Ability/method	unknown-variance t confidence interval; K36

Course-method solution. Use $(\bar{X}-\mu)/(S/\sqrt{n})\sim t_{14}$. Here $s=2.5$. The margin is $2.145(2.5/\sqrt{15})=1.385$, giving $(72.815, 75.585)$.

Marking points. 1 mark: t pivot and df; 1 mark: formula; 1 mark: margin; 1 mark: interval

Common errors. Using $z=1.96$ or $s^2/\sqrt{n}$.

Final self-check. The interval is centered at 74.2 and has symmetric endpoints.

Difficulty allocation: A:2, B:2; estimated normal time 6 min.

3(d) (5 marks)

Recognition signal	The same t pivot used for the interval becomes the test statistic.
First key step	Standardize the observed mean under H_0 .
Historical basis	2021 Q5(b)(i)-(iii); 2026 Q3(d)(ii)
Ability/method	one-sample t test and p-value; K39

Course-method solution. $T=(74.2-75)/(2.5/\sqrt{15})=-1.239$ with 14 df. The two-sided p-value is $2P(t_{14}\geq 1.239)=0.236$ (approximately). Since $p>0.05$, fail to reject H_0 ; this is not proof that $\mu=75$.

Marking points. 1 mark: hypotheses/test; 1 mark: statistic; 1 mark: df; 1 mark: p-value; 1 mark: conclusion

Common errors. One-sided p-value; saying accept H_0 .

Final self-check. The null value 75 lies inside the 95% interval, consistent with non-rejection.

Difficulty allocation: A:2, B:1, C:2; estimated normal time 8 min.

3(e) (5 marks)

Recognition signal	The question asks for both the random-variable property under H_0 and the observed-data interpretation.
First key step	State the tail event relative to the null distribution.
Historical basis	2021 Q5(d); 2024 Q5(a); 2026 Q3(e)
Ability/method	p-value definition, interpretation and null distribution; K41

Course-method solution. A p-value is the null probability of obtaining a test statistic at least as incompatible with H_0 as the observed value, using the pre-specified tail rule. Small values are evidence against H_0 , not the probability that H_0 is true. If the null distribution of a continuous test statistic is correctly specified (or the p-value uses a continuous null CDF), the p-value is $\text{Uniform}(0,1)$ under H_0 .

Marking points. 2 marks: definition; 1 mark: interpretation; 1 mark: uniformity conditions; 1 mark: limitation

Common errors. Calling it $P(H_0 \text{ true}|\text{data})$; omitting continuity/exactness.

Final self-check. Under H_0 , $P(p \leq \alpha) = \alpha$ for exact continuous p-values.

Difficulty allocation: A:2, B:1, C:2; estimated normal time 6 min.

Total: 20 marks

Question 4

4(a) (4 marks)

Recognition signal	Correlation of a linear combination is found from covariance bilinearity and variance expansion.
First key step	Convert correlation to covariance first.
Historical basis	2023 Q5(c); 2026 Q4(a)
Ability/method	covariance algebra for linear combinations; K27

Course-method solution. $\text{Cov}(X,Y)=(-1/3)(3)(2)=-2$. $\text{Cov}(X,Z)=\text{Var}(X)-2\text{Cov}(X,Y)=9+4=13$. $\text{Var}(Z)=9+4(4)-4\text{Cov}(X,Y)=33$. Hence $\text{Corr}(X,Z)=13/(3\sqrt{33})$.

Marking points. 1 mark: covariance; 1 mark: $\text{Cov}(X,Z)$; 1 mark: $\text{Var}(Z)$; 1 mark: correlation

Common errors. Using means; forgetting the cross term in $\text{Var}(Z)$.

Final self-check. The absolute value $13/(3\sqrt{33})$ is below 1.

Difficulty allocation: A:2, B:2; estimated normal time 6 min.

4(b) (5 marks)

Recognition signal	The target is uncorrelatedness, so compute one covariance directly.
First key step	Expand $\text{Cov}(X+Y,X-Y)$ term by term.
Historical basis	2024 Q5(c)
Ability/method	proof by covariance bilinearity and equal-variance cancellation; K27

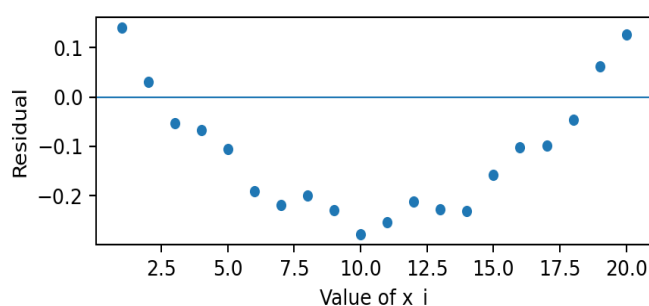
Course-method solution. By bilinearity and symmetry of covariance, $\text{Cov}(X+Y,X-Y)=\text{Var}(X)-\text{Cov}(X,Y)+\text{Cov}(Y,X)-\text{Var}(Y)=\text{Var}(X)-\text{Var}(Y)=0$. Therefore $X+Y$ and $X-Y$ are uncorrelated. Independence of X and Y is not required.

Marking points. 1 mark: expand covariance; 2 marks: use symmetry/cancellation; 1 mark: equal variances; 1 mark: conclusion

Common errors. Assuming X and Y are independent; confusing uncorrelatedness with independence.

Final self-check. The cross-covariance terms cancel and only the variance difference remains.

Difficulty allocation: A:1, B:1, C:1, D:2; estimated normal time 7 min.



4(c) (4 marks)

Recognition signal	Residual structure overrides a superficially high R^2 .
First key step	Look for randomness around zero and constant spread.
Historical basis	2021 Q6(c); 2025 Q4(d); 2026 Q4(b)
Ability/method	residual diagnosis and transformation; K50

Course-method solution. No. The U-shaped residual pattern shows systematic curvature, so the linear mean structure is inadequate despite high R^2 . Try a transformed predictor such as x^2 or fit a polynomial/appropriate transformation, then recheck residuals.

Marking points. 1 mark: reject fit; 1 mark: identify pattern; 1 mark: R^2 limitation; 1 mark: specific remedy

Common errors. Saying the fit is good solely because R^2 is high.

Final self-check. After a remedy, residuals should show no systematic pattern.

Difficulty allocation: A:1, C:1, D:2; estimated normal time 5 min.

4(d) (5 marks)

Recognition signal	A support-dependent likelihood may be flat and non-unique.
First key step	Write the support inequalities before differentiating; no derivative is useful.
Historical basis	2026 Q4(f)
Ability/method	support-dependent non-unique MLE; K54

Course-method solution. The likelihood is 2^{-n} when $\theta \leq \min x_i$ and $\theta \geq \max x_i - 2$, and zero otherwise. Thus every θ in $[\max x_i - 2, \min x_i]$ is an MLE; the likelihood is flat on that interval.

Marking points. 1 mark: individual density/support; 1 mark: joint likelihood; 2 marks: convert all inequalities; 1 mark: full MLE set

Common errors. Forcing a unique endpoint; ignoring the zero-likelihood region.

Final self-check. The interval is nonempty exactly when the observed range is at most 2.

Difficulty allocation: C:1, D:4; estimated normal time 8 min.

4(e) (2 marks)

Recognition signal	A bootstrap resample is sampled with replacement and has the same size.
First key step	Check size and membership, not uniqueness.
Historical basis	2022 Q6(b)(i)-(ii); 2024 Q6(e)(i)-(ii); 2026 Q4(e)(i)-(ii)
Ability/method	bootstrap sample definition; K52

Course-method solution. (i) Yes: it has size 5 and every value comes from the original sample; repetition is allowed. (ii) No: it has only 4 observations, whereas a nonparametric bootstrap resample has the original size.

Marking points. 1 mark each classification with the decisive rule

Common errors. Rejecting repetitions; accepting a shorter sample.

Final self-check. A valid sample can repeat values but cannot introduce new values.

Difficulty allocation: A:1, B:1; estimated normal time 3 min.

Total: 20 marks