

MATH40005

Probability and Statistics

Revised Mock Examination 4 - Solutions and Marking Guide

Time allowed	2 hours
Questions	4
Total marks	80
Format	Closed book; justify all answers
Difficulty calibration	A 32, B 20, C 12, D 16
Paper role	2020-2022 core-method paper: definitions, stars-and-bars, double counting, geometric waits, estimator risk, inequalities, OLS, support likelihood and conjugacy.

- Answer all four questions.
- Start each question on a new page in an actual timed attempt.
- State assumptions and named results used.
- Statistical tables or stated critical values may be used where supplied.

Question 1

1(a) (3 marks)

Recognition signal	A definition question requires all closure axioms.
First key step	List the three axioms.
Historical basis	2020 Q1(a)
Ability/method	sigma-algebra definition; K02

Course-method solution. F is a collection of subsets of Ω containing the empty set, closed under complements, and closed under countable unions.

Marking points. 1 mark each axiom

Common errors. Saying only “all events”; forgetting countable union.

Final self-check. The axioms imply Ω is in F and closure under countable intersections.

Difficulty allocation: A:3; estimated normal time 3 min.

1(b) (3 marks)

Recognition signal	A probability measure is more than a function assigning numbers.
First key step	State non-negativity, normalization and countable additivity.
Historical basis	2020 Q1(b)
Ability/method	probability-measure definition; K03

Course-method solution. $P:F \rightarrow [0,1]$ is non-negative, satisfies $P(\Omega)=1$, and is countably additive on pairwise disjoint events.

Marking points. 1 mark each axiom

Common errors. Using finite additivity only.

Final self-check. It follows that $P(\text{empty})=0$.

Difficulty allocation: A:3; estimated normal time 3 min.

1(c) (4 marks)

Recognition signal	Positive ordered parts correspond to bars in gaps, not unrestricted bins.
First key step	Place $n-1$ bars among $k-1$ gaps.
Historical basis	2020 Q1(e)
Ability/method	positive stars and bars; K06

Course-method solution. If $k < n$ there are none. If $k \geq n$, represent k stars and choose $n-1$ of the $k-1$ internal gaps for bars, giving $C(k-1, n-1)$.

Marking points. 1 mark impossible case; 2 marks stars-and-bars construction; 1 mark formula

Common errors. Using $C(k+n-1, n-1)$, which allows zeros.

Final self-check. For $k=n$ the formula gives 1.

Difficulty allocation: A:2, B:1, D:1; estimated normal time 5 min.

1(d) (4 marks)

Recognition signal	The factor $n-k$ suggests converting $C(n,k)(n-k)$ to $nC(n-1,k)$.
First key step	Start from the left and expand $(3+1)^{n-1}$.
Historical basis	2022 Q1(b)(i)
Ability/method	binomial identity proof; K05

Course-method solution. Use $nC(n-1,k) = C(n,k)(n-k)$. Then $n4^{n-1} = n(3+1)^{n-1} = \sum_{k=0}^{n-1} nC(n-1,k)3^k = \sum_{k=0}^{n-1} C(n,k)3^k(n-k)$, with the $k=n$ term zero.

Marking points. 1 mark binomial theorem; 2 marks coefficient identity; 1 mark endpoint

Common errors. Differentiating incorrectly or losing the zero term.

Final self-check. Check $n=1$: both sides equal 1.

Difficulty allocation: A:1, B:2, C:1; estimated normal time 6 min.

1(e) (6 marks)

Recognition signal	A story proof requires two counts of exactly the same object.
First key step	Interpret $n-k$ as the choice of a special person among non-alcohol drinkers.
Historical basis	2022 Q1(b)(ii)
Ability/method	double counting/story proof; K05

Course-method solution. Count drink orders for n people with four choices (three alcoholic and water) together with a designated sober driver who drinks water. Left: choose the driver (n ways), then each other person has four choices, $n4^{\{n-1\}}$. Right: choose k people who take an alcoholic drink and choose one of 3 drinks for each, $C(n,k)3^k$; choose the driver among the remaining $n-k$ water drinkers. Sum over k .

Marking points. 2 marks common combinatorial object; 2 marks left count; 2 marks right count

Common errors. Giving two unrelated stories; not explaining the sum over k .

Final self-check. Both counts include a designated driver and all drink assignments.

Difficulty allocation: B:2, C:1, D:3; estimated normal time 9 min.

Total: 20 marks

Question 2

2(a) (4 marks)

Recognition signal	A discrete CDF is a right-continuous step function.
First key step	Accumulate masses in support order.
Historical basis	2020 Q2(b)
Ability/method	piecewise discrete CDF; K11

Course-method solution. $F_X(x)=0$ for $x<1$; $1/2$ for $1\leq x<3$; $5/6$ for $3\leq x<5$; and 1 for $x\geq 5$.

Marking points. 1 mark each interval

Common errors. Defining it only at 1,3,5.

Final self-check. Jumps equal the pmf and the final value is 1.

Difficulty allocation: A:4; estimated normal time 4 min.

2(b) (6 marks)

Recognition signal	A minimum is easiest through the event both variables exceed k.
First key step	Use survival functions.
Historical basis	2020 Q2(c)(ii)
Ability/method	minimum of independent geometric waits; K14,K04

Course-method solution. For integer $k\geq 0$, $P(\min(X,Y)>k)=P(X>k)P(Y>k)=[(1-p_1)(1-p_2)]^k$. Thus the minimum is $\text{Geom}(q)$ on $\{1,2,\dots\}$ with $q=1-(1-p_1)(1-p_2)$.

Marking points. 1 mark survival function; 1 mark independence; 2 marks algebra; 2 marks identify parameter/support

Common errors. Multiplying CDFs rather than survivals.

Final self-check. If p_1 or p_2 increases, q should increase.

Difficulty allocation: A:2, B:2, C:1, D:1; estimated normal time 7 min.

2(c) (6 marks)

Recognition signal	The target pattern can be decomposed into first reaching H, then reaching T while a trailing H remains available.
First key step	Use a state or two-stage waiting argument.
Historical basis	2020 Q2(d)
Ability/method	pattern waiting by states/geometric stages; K14,K23

Course-method solution. Wait for the first H: a $\text{Geom}(1/2)$ time with mean 2. After an H has occurred, wait for the first subsequent T; further H outcomes preserve a trailing H, so this is another $\text{Geom}(1/2)$ wait with mean 2. The total expected time is 4.

Marking points. 2 marks decomposition; 2 marks geometric means; 2 marks total/justification

Common errors. Treating non-overlapping pairs as independent trials.

Final self-check. A short Markov-state recursion gives the same answer 4.

Difficulty allocation: A:1, B:1, C:1, D:3; estimated normal time 8 min.

2(d) (4 marks)

Recognition signal	Partition conditioning expands each mass by total probability.
First key step	Start from the definition of expectation.
Historical basis	2020 Q3(d)(ii)
Ability/method	proof of total expectation; K23,K03

Course-method solution. $E[X]=\sum_x xP(X=x)=\sum_x x \sum_i P(X=x|B_i)P(B_i)$. Absolute convergence allows interchange: $=\sum_i P(B_i)\sum_x xP(X=x|B_i)=\sum_i E[X|B_i]P(B_i)$.

Marking points. 1 mark total probability; 1 mark interchange justification; 2 marks conclusion

Common errors. Interchanging infinite sums without the stated condition.

Final self-check. For a two-set partition it reduces to the familiar formula.

Difficulty allocation: A:1, B:2, C:1; estimated normal time 6 min.

Total: 20 marks

Question 3

3(a) (6 marks)

Recognition signal	A scaled sample sum requires separate expectation and variance calculations.
First key step	Compute E and Var before applying bias-variance.
Historical basis	2022 Q4(a)(i)-(ii); 2020 Q4(a)(iv)-(v)
Ability/method	bias/variance/MSE of an estimator; K31

Course-method solution. $E[T] = n\theta/(n+1)$, so $\text{Bias} = -\theta/(n+1)$. $\text{Var}(T) = n\sigma^2/(n+1)^2$. Hence $\text{MSE} = (\theta^2 + n\sigma^2)/(n+1)^2$.

Marking points. 2 marks bias; 2 marks variance; 2 marks MSE

Common errors. Using $n^2\sigma^2$; omitting squared bias.

Final self-check. MSE is non-negative and decreases at order $1/n$ for fixed parameters.

Difficulty allocation: A:3, B:2, C:1; estimated normal time 7 min.

3(b) (4 marks)

Recognition signal	The event indicator converts a tail probability into an expectation bound.
First key step	Use $X \geq a \implies \mathbb{1}\{X \geq a\}$.
Historical basis	2020 Q4(b)
Ability/method	indicator proof of Markov inequality; K28

Course-method solution. For $a > 0$, $X \geq a \implies \mathbb{1}\{X \geq a\}$. Taking expectations gives $E[X] \geq aP(X \geq a)$, hence $P(X \geq a) \leq E[X]/a$.

Marking points. 1 mark pointwise inequality; 2 marks expectation; 1 mark result

Common errors. Assuming a density; Markov is distribution-free.

Final self-check. Equality can occur for a variable taking only 0 and a .

Difficulty allocation: A:2, B:2; estimated normal time 5 min.

3(c) (5 marks)

Recognition signal	Uniformity in p calls for $p(1-p) \leq 1/4$.
First key step	Bound the worst-case variance before solving for n .
Historical basis	2022 Q4(b); 2023 Q4(b)
Ability/method	distribution-free sample-size guarantee; K45, K28

Course-method solution. $\text{Var}(\hat{p}) = p(1-p)/n \leq 1/(4n)$. Chebyshev gives $P(|\hat{p} - p| \geq 0.02) \leq 1/[4n(0.02)^2]$. Require this ≤ 0.05 , so $n \geq 12500$.

Marking points. 1 mark variance; 1 mark uniform bound; 2 marks Chebyshev inequality; 1 mark n

Common errors. Using a normal approximation without permission.

Final self-check. Substitution $n=12500$ makes the upper error bound exactly 0.05.

Difficulty allocation: A:1, B:1, C:1, D:2; estimated normal time 8 min.

3(d) (5 marks)

Recognition signal	Multiple tests require controlling the family-wise error, not comparing each to 0.01.
First key step	Divide α by the number of tests.
Historical basis	2020 Q4(c); 2022 Q5(c)
Ability/method	Bonferroni correction; K43

Course-method solution. The adjusted threshold is $0.01/100 = 0.0001$. Only 0.00002 and 0.00005 are below it.

Marking points. 2 marks threshold; 2 marks comparisons; 1 mark conclusion

Common errors. Using 0.01 for each test.

Final self-check. Exactly two values are below 10^{-4} .

Difficulty allocation: A:2, B:1, C:1, D:1; estimated normal time 5 min.

Total: 20 marks

Question 4

4(a) (6 marks)

Recognition signal	A quadratic RSS can be minimized by derivatives or completing the square.
First key step	Eliminate the intercept using the sample means.
Historical basis	2020 Q5(b); 2023 Q6(b)
Ability/method	OLS derivation; K48

Course-method solution. Minimize RSS. For fixed β_1 , the minimizing β_0 is $\bar{y} - \beta_1 \bar{x}$. Substitution gives $RSS = S_{yy} - 2\beta_1 S_{xy} + \beta_1^2 S_{xx}$, minimized at $\hat{\beta}_1 = S_{xy}/S_{xx}$. Thus $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$.

Marking points. 2 marks intercept condition; 2 marks reduced RSS; 1 mark slope; 1 mark intercept

Common errors. Writing formulas without proving minimization.

Final self-check. The fitted line passes through $(\bar{x}, \bar{y})$.

Difficulty allocation: A:3, B:2, C:1; estimated normal time 9 min.

4(b) (5 marks)

Recognition signal	The parameter determines the support, so the maximum occurs at a boundary.
First key step	State the zero-likelihood region.
Historical basis	2020 Q5(d)
Ability/method	boundary MLE; K54

Course-method solution. $L(\theta) = \theta^{-n}$ if $\theta \geq x_{\max}$ and all data are non-negative, otherwise 0. Since θ^{-n} decreases, the smallest feasible θ maximizes it: $\hat{\theta} = x_{\max}$.

Marking points. 2 marks likelihood/support; 2 marks monotonicity/boundary; 1 mark estimator

Common errors. Differentiating $\log L = -n \log \theta$ and setting it to zero.

Final self-check. The estimate must be at least every observation.

Difficulty allocation: A:1, B:1, C:1, D:2; estimated normal time 7 min.

4(c) (4 marks)

Recognition signal	Conjugacy is recognized by matching the gamma kernel.
First key step	Ignore constants independent of θ and collect powers/rates.
Historical basis	2020 Q6(c); 2022 Q6(e) related conjugacy method
Ability/method	Gamma conjugate update; K55

Course-method solution. The likelihood is $\theta^n \exp(-\theta \sum x_i)$. Multiplying by $\theta^{a-1} \exp(-b\theta)$ gives posterior kernel $\theta^{a+n-1} \exp[-(b + \sum x_i)\theta]$. Thus $\theta|x \sim \text{Gamma}(a+n, b + \sum x_i)$.

Marking points. 1 mark likelihood; 1 mark prior product; 1 mark kernel; 1 mark identified posterior

Common errors. Using scale instead of stated rate.

Final self-check. Posterior shape and rate both increase by data information.

Difficulty allocation: A:1, B:1, C:1, D:1; estimated normal time 6 min.

4(d) (5 marks)

Recognition signal	A covariance plus bounded Y can be converted into a variance bound.
First key step	Use Cauchy/correlation first, then the range variance bound.
Historical basis	2020 Q6(d)
Ability/method	non-trivial variance lower bound; K33, K27, K28

Course-method solution. $\text{Cov}(X, Y) = 4 - 6 = -2$. Since $\text{Corr}^2 \leq 1$, $\text{Var}(X) \geq \text{Cov}^2 / \text{Var}(Y) = 4 / \text{Var}(Y)$. For Y in $[2, 5]$, $\text{Var}(Y) \leq (5-2)^2 / 4 = 9/4$. Hence $\text{Var}(X) \geq 16/9$ and $\text{sd}(X) \geq 4/3$.

Marking points. 1 mark covariance; 1 mark correlation bound; 1 mark bounded-variance result; 2 marks combine

Common errors. Using $\text{Var}(Y) \geq 9/4$; the inequality direction is opposite.

Final self-check. Squaring the final bound gives $16/9$.

Difficulty allocation: A:2, C:1, D:2; estimated normal time 8 min.

Total: 20 marks