

MATH40005

Probability and Statistics

Revised Mock Examination 5 - Solutions and Marking Guide

Time allowed	2 hours
Questions	4
Total marks	80
Format	Closed book; justify all answers
Difficulty calibration	A 32, B 20, C 12, D 16
Paper role	Seven-year gap paper: measurability, preimage sigma-algebras, scale mixtures, mixed transformations, exact tests, quartiles, bounds and estimator construction.

- Answer all four questions.
- Start each question on a new page in an actual timed attempt.
- State assumptions and named results used.
- Statistical tables or stated critical values may be used where supplied.

Question 1

1(a) (2 marks)

Recognition signal	Indicator identities reduce to membership equivalence.
First key step	Write the two “equals 1 iff” statements.
Historical basis	2023 Q2(a)(i)
Ability/method	indicator/preimage identity; K02

Course-method solution. Both sides equal 1 exactly when $f(u)$ belongs to A , equivalently u belongs to $f^{-1}(A)$, and equal 0 otherwise.

Marking points. 2 marks equivalence by cases

Common errors. Confusing image and preimage.

Final self-check. Test a point inside and outside the preimage.

Difficulty allocation: A:2; estimated normal time 3 min.

1(b) (6 marks)

Recognition signal	Preimages preserve set operations exactly.
First key step	Verify the three sigma-algebra axioms using preimage identities.
Historical basis	2023 Q2(a)(ii)
Ability/method	preimage sigma-algebra proof; K02

Course-method solution. Preimages satisfy $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(A^c) = [f^{-1}(A)]^c$, and $f^{-1}(\cup A_i) = \cup f^{-1}(A_i)$. Since G has the corresponding closure properties, the collection of preimages is a sigma-algebra.

Marking points. 1 mark empty set; 2 marks complements; 2 marks countable unions; 1 mark conclusion

Common errors. Trying to prove f is one-to-one; it is unnecessary.

Final self-check. No probability measure is needed.

Difficulty allocation: A:2, B:2, D:2; estimated normal time 8 min.

1(c) (4 marks)

Recognition signal	The standard parity construction separates pairwise from joint independence.
First key step	Use two independent bits and their xor.
Historical basis	2026 Q2(a)
Ability/method	pairwise-versus-mutual independence construction; K04

Course-method solution. Let U, V be independent Bernoulli($1/2$), and $W = U \text{ xor } V$. Each pair is uniform on $\{0,1\}^2$, hence independent. But W is determined by U, V , and $P(U=0, V=0, W=0) = 1/4$ differs from $(1/2)^3$.

Marking points. 1 mark construction; 2 marks pairwise checks; 1 mark failure of mutual independence

Common errors. Giving three mutually independent variables.

Final self-check. The triple has only four possible outcomes, not eight.

Difficulty allocation: A:2, B:1, C:1; estimated normal time 5 min.

1(d) (4 marks)

Recognition signal	Finite measurability is checked by singleton preimages.
First key step	Test the preimage of 0.
Historical basis	2020 Q2(a); 2022 Q1(a)(ii)
Ability/method	finite-space measurability check; K02

Course-method solution. No, because $X^{-1}(\{0\}) = \{1,3\}$ is not in $\mathcal{F}$. A valid example is $Y=0$ on $\{1,2\}$ and $Y=1$ on $\{3,4\}$; its singleton preimages belong to $\mathcal{F}$.

Marking points. 2 marks counterexample preimage; 2 marks valid example/justification

Common errors. Checking only the image values.

Final self-check. A measurable variable must be constant on each atom $\{1,2\}$ and $\{3,4\}$.

Difficulty allocation: A:2, B:1, C:1; estimated normal time 5 min.

1(e) (4 marks)

Recognition signal	Repeated finite trials produce a Cartesian product; a finite power set has size $2^{ \Omega }$.
First key step	Count outcomes before subsets.
Historical basis	2021 Q1(b)(i)-(iii)
Ability/method	product probability space and power set; K01,K05

Course-method solution. $\Omega = \{1, \dots, 6\}^7$, $|\Omega| = 6^7$, $\mathcal{F} = \mathcal{P}(\Omega)$, $P(A) = |A|/6^7$. The power set has cardinality 2^{6^7} .

Marking points. 1 mark sample space; 1 mark probability space; 2 marks power-set cardinality

Common errors. Answering 2^7 or 6^{2^7} .

Final self-check. The event-space cardinality is vastly larger than the sample-space cardinality.

Difficulty allocation: A:2, B:2; estimated normal time 5 min.

Total: 20 marks

Question 2

2(a) (7 marks)

Recognition signal	A random scale mixture signals conditional expectation.
First key step	Condition on the scale variable Y.
Historical basis	2021 Q2(d)
Ability/method	MGF of a scale mixture; K12,K23,K24

Course-method solution. Condition on $Y=y$: $E[e^{uZ}|Y=y]=M_X(u\sqrt{y})=\exp(u^2 y/2)$. Thus $M_Z(u)=M_Y(u^2/2)=[\beta/(\beta-u^2/2)]^\alpha$, requiring $u^2/2<\beta$, i.e. $|u|<\sqrt{2\beta}$.

Marking points. 2 marks conditioning; 1 mark normal MGF; 2 marks gamma MGF; 1 mark domain; 1 mark result

Common errors. Replacing $E[\exp(u^2 Y/2)]$ by $\exp(u^2 E[Y]/2)$.

Final self-check. At $u=0$ the MGF is 1.

Difficulty allocation: A:1, B:1, C:2, D:3; estimated normal time 10 min.

2(b) (6 marks)

Recognition signal	A product with a positive discrete multiplier is a finite mixture of scaled densities.
First key step	Condition on Y and use the CDF method.
Historical basis	2022 Q2(c)(i)
Ability/method	continuous-discrete mixture transform; K15,K20,K23

Course-method solution. Conditioning on Y, $F_Z(z)=\sum_{y=1}^{10} F_X(z/y)P(Y=y)$. Differentiating gives $f_Z(z)=\sum_{y=1}^{10} [1/y]f_X(z/y)P(Y=y)$. It is non-negative and integrates to $\sum_y P(Y=y)=1$ after the change $v=z/y$.

Marking points. 1 mark condition/CDF; 2 marks differentiation; 1 mark non-negativity; 2 marks integral check

Common errors. Omitting the Jacobian $1/y$.

Final self-check. Each component density integrates to its mixture weight.

Difficulty allocation: A:2, B:2, D:2; estimated normal time 9 min.

2(c) (7 marks)

Recognition signal	An ordered simplex has nested bounds.
First key step	Normalize first, then reuse the same bounds for the moment.
Historical basis	2020 Q3(c)
Ability/method	three-dimensional normalization and LOTUS; K17,K22,K20

Course-method solution. The ordered region has volume $1/6$, so $c=6$. Direct LOTUS gives $E[XYZ]=6 \int_0^1 \int_0^z \int_0^y xyz \, dx \, dy \, dz=1/8$.

Marking points. 2 marks normalization/constant; 3 marks triple integral; 2 marks result

Common errors. Assuming X,Y,Z independent.

Final self-check. The order-statistic symmetry gives $c=3!=6$.

Difficulty allocation: A:2, B:1, C:1, D:3; estimated normal time 10 min.

Total: 20 marks

Question 3

3(a) (5 marks)

Recognition signal	Fixed-size classification without replacement gives a hypergeometric null.
First key step	Use the upper tail including the observed result.
Historical basis	2024 Q5(b) variant
Ability/method	exact hypergeometric test; K10,K44

Course-method solution. Under H_0 , $X \sim \text{Hypergeometric}(10,5,5)$. The p-value $P(X \geq 4) = [C(5,4)C(5,1) + C(5,5)C(5,0)]/C(10,5) = 26/252 = 13/126 \approx 0.103$. Fail to reject H_0 .

Marking points. 1 mark model/hypotheses; 2 marks tail; 1 mark value; 1 mark conclusion

Common errors. Using only $P(X=4)$.

Final self-check. The p-value exceeds 0.05.

Difficulty allocation: A:2, B:1, C:1, D:1; estimated normal time 7 min.

3(b) (4 marks)

Recognition signal	Tukey outliers are based on quartiles and 1.5 IQR fences.
First key step	Sort before splitting the sample.
Historical basis	2023 Q4(d)
Ability/method	quartiles and Tukey outliers; K46

Course-method solution. Sorted: 6.1,7.2,8.3,9.4,10.5,11.6,12.7,14.3,15.9,24.0. Lower half median $q_1=8.3$; upper half median $q_3=14.3$. IQR=6.0, fences $[-0.7,23.3]$, so 24.0 is the only outlier.

Marking points. 1 mark sorting/quartiles; 1 mark IQR; 1 mark fences; 1 mark outlier

Common errors. Including both middle observations in halves for even n .

Final self-check. The identified value must lie outside a computed fence.

Difficulty allocation: A:2, B:1, C:1; estimated normal time 6 min.

3(c) (3 marks)

Recognition signal	A reversal after aggregation is the signature of Simpson's paradox.
First key step	Compare subgroup weights and baseline risks.
Historical basis	2021 Q6(d); 2023 Q5(d)
Ability/method	Simpson reversal/confounding; K51

Course-method solution. This is Simpson's paradox. It occurs when treatment allocation is unbalanced across groups with different baseline risks; aggregation changes the weights and can reverse the within-group comparison. The risk group is a confounder.

Marking points. 1 mark name; 1 mark unequal weighting; 1 mark contextual explanation

Common errors. Calling it random error without explaining the weighting mechanism.

Final self-check. Stratified comparisons preserve the within-group direction.

Difficulty allocation: A:1, B:1, C:1; estimated normal time 4 min.

3(d) (5 marks)

Recognition signal	No distribution is specified, so use the variance bound rather than normal theory.
First key step	Translate confidence and width separately.
Historical basis	2021 Q4(b)
Ability/method	Chebyshev sample-size design; K45,K28

Course-method solution. Since only the bound $\sigma < 10$ is known, report the interval $\bar{X} \pm 10/\sqrt{0.01n} = \bar{X} \pm 100/\sqrt{n}$. Chebyshev gives coverage at least 0.99. Its total width is $200/\sqrt{n}$. The strict requirement $200/\sqrt{n} < 4$ is equivalent to $n > 2500$. Hence the minimum integer is $n = 2501$.

Marking points. 1 mark use known bound 10; 1 mark half-width $100/\sqrt{n}$; 1 mark total width; 1 mark strict inequality $n > 2500$; 1 mark minimum integer $n = 2501$.

Common errors. Using the unknown actual sigma in a reported interval; accepting $n = 2500$, for which the bound-based width is exactly 4; using z critical values.

Final self-check. $200/\sqrt{2501} < 4$, whereas $200/\sqrt{2500} = 4$.

Difficulty allocation: A:1, B:1, C:1, D:2; estimated normal time 8 min.

3(e) (3 marks)

Recognition signal	The two errors condition on different truths.
First key step	State the decision and the true state.
Historical basis	2022 Q5(a); 2025 Q5(c)
Ability/method	testing error trade-off; K42

Course-method solution. Type I error is rejecting a true H_0 , with probability α for a level- α test. Type II error is failing to reject a false H_0 . With other design features fixed, increasing α enlarges the rejection region, increasing Type I error and generally decreasing Type II error/increasing power.

Marking points. 1 mark each definition; 1 mark trade-off

Common errors. Saying both decrease when α increases.

Final self-check. A larger rejection region makes rejection more frequent under both null and alternative.

Difficulty allocation: A:1, B:1, C:1; estimated normal time 4 min.

Total: 20 marks

Question 4

4(a) (6 marks)

Recognition signal	Known-variance lognormal location estimation becomes least squares on log data.
First key step	Discard theta-free terms and differentiate.
Historical basis	2022 Q6(a)
Ability/method	regular log-likelihood MLE; K53

Course-method solution. The log-likelihood in θ is constant $-[1/(2\sigma^2)]\sum(\log x_i - \theta)^2$. Differentiating gives $\sum(\log x_i - \theta) = 0$, so $\hat{\theta} = n^{-1}\sum \log x_i$. The second derivative is $-n/\sigma^2 < 0$.

Marking points. 2 marks log-likelihood; 2 marks score equation; 1 mark maximum check; 1 mark estimator

Common errors. Using the arithmetic mean of x_i .

Final self-check. The estimate is the sample mean of transformed observations.

Difficulty allocation: A:2, B:1, C:1, D:2; estimated normal time 8 min.

4(b) (4 marks)

Recognition signal	The intercept normal equation forces mean residual zero.
First key step	Differentiate RSS with respect to β_0 .
Historical basis	2022 Q6(d)
Ability/method	OLS mean-point property; K48

Course-method solution. The normal equation for the intercept is $\sum(y_i - \beta_0 - \beta_1 x_i) = 0$, so $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$. Hence at $x = \bar{x}$, $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x} = \bar{y}$.

Marking points. 2 marks normal equation; 1 mark intercept relation; 1 mark point conclusion

Common errors. Using the result without justification.

Final self-check. The fitted residuals sum to zero.

Difficulty allocation: A:3, B:1; estimated normal time 5 min.

4(c) (4 marks)

Recognition signal	Moment equality requires a determinacy condition.
First key step	Use MGF existence near zero.
Historical basis	2025 Q3(c)(i)
Ability/method	moment determinacy; K34, K12

Course-method solution. If both MGFs exist on an open interval around 0, their Taylor expansions have the same moment coefficients, so the MGFs agree there. MGF uniqueness yields equal distributions.

Marking points. 1 mark condition; 2 marks MGF equality; 1 mark uniqueness

Common errors. Claiming the conclusion without a condition.

Final self-check. MGFs that agree near zero uniquely determine distributions.

Difficulty allocation: A:1, B:1, C:1, D:1; estimated normal time 5 min.

4(d) (3 marks)

Recognition signal	The bootstrap distribution is the distribution of replicate medians.
First key step	Resample with replacement at the original size.
Historical basis	2025 Q6(c)
Ability/method	bootstrap percentile interval; K52

Course-method solution. Generate many size- n resamples with replacement, compute each median, and take the empirical 2.5th and 97.5th percentiles.

Marking points. 1 mark resampling; 1 mark statistic; 1 mark percentiles

Common errors. Taking percentiles of raw data.

Final self-check. The endpoints come from bootstrap medians.

Difficulty allocation: A:2, B:1; estimated normal time 4 min.

4(e) (3 marks)

Recognition signal	Correct the upward bias of $\bar{X}^2$ using an unbiased variance estimate.
First key step	Compute $E[\bar{X}^2]$.
Historical basis	2021 Q4(a)(iii)
Ability/method	constructing an unbiased estimator; K32

Course-method solution. Since $E[\bar{X}^2] = \mu^2 + \sigma^2/n$ and $E[S^2] = \sigma^2$, the estimator $\bar{X}^2 - S^2/n$ is unbiased for μ^2 .

Marking points. 1 mark moment identity; 1 mark estimator; 1 mark unbiasedness

Common errors. Using $\bar{X}^2$ alone.

Final self-check. Taking expectations returns exactly μ^2 .

Difficulty allocation: A:2, B:1; estimated normal time 5 min.

Total: 20 marks