

MATH40007 - An Introduction to Applied Mathematics

Revised Practice Paper 1

2026 structure reinforcement

Time allowed: 2 hours
 Total marks: 80
 Questions: Answer all four questions
 Historical role: One controlled paper close to 2026; not a claim of seven-year coverage

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

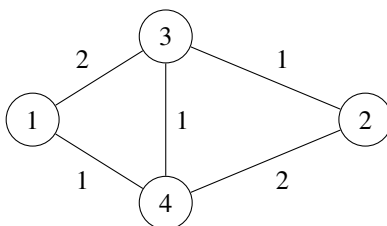
Question 1

(a) Consider

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{pmatrix}.$$

Find the rank of M and give bases for its column space, right null space, row space and left null space. (8 marks)

(b) In the circuit below, node 1 is held at voltage 1 and node 2 is grounded. Edge labels are conductances and Kirchhoff's current law holds at nodes 3 and 4.

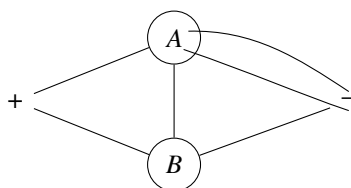


Find the voltages at nodes 3 and 4, the current in edge 3–4 including its direction, and the effective conductance between nodes 1 and 2. (12 marks) (Total: 20 marks)

Question 2

All edges in the diagrams below have unit conductance. Parallel edges are distinct edges.

- (a) The first network has terminals $+$ and $-$ and internal nodes A, B . The edges are $+A$, $+B$, two parallel edges $A-$, one edge $B-$ and one edge AB .



Find its effective conductance.

(8 marks)

- (b) A second network consists of two parallel paths from $+$ to $-$, one of length 2 and one of length 4. Find its effective conductance.

(6 marks)

- (c) Put the two networks in series, separated by one extra unit edge. A random walker starts at the first $+$, chooses uniformly among incident edges and stops on reaching the final $-$ or returning to its starting point. Find the probability of reaching $-$ first.

(6 marks)

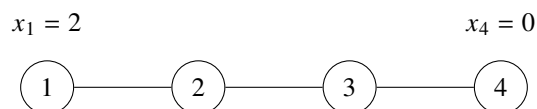
(Total: 20 marks)

Question 3

Diffusion takes place on the line graph 1 – 2 – 3 – 4. Boundary concentrations are fixed at

$$x_1(t) = 2, \quad x_4(t) = 0,$$

there are no sources at nodes 2 and 3, and $x_2(0) = 0, x_3(0) = 1$.



Let $\hat{x} = (x_2, x_3)^T$ and $K_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

- (a) Derive the reduced differential equation, find the steady state and give an orthonormal eigensystem of K_2 . (8 marks)
- (b) Find explicit formulae for $x_2(t)$ and $x_3(t)$. (8 marks)
- (c) State the initial and limiting values and describe a correct sketch. (4 marks) (Total: 20 marks)

Question 4

A uniform rod occupies $0 \leq x \leq 1$. Its electrical conductivity is $\sigma = 2$ and thermal conductivity is $k = 3$.

(a) The endpoints are held at $\phi(0) = 0$, $\phi(1) = 1$, with no interior current sources. Find ϕ , the electrical current density $J = -\sigma\phi'$ and the electrical dissipation density $q_e = -\phi'J$. The ends are then held at temperatures $T(0) = 0$, $T(1) = 1$, and all electrical dissipation becomes heat. Find the steady temperature. (10 marks)

(b) Replace the thermal condition at $x = 0$ by insulation, retaining $T(1) = 1$. Find the new temperature and the outward heat flux at $x = 1$. Verify the answer by an energy balance. (10 marks) (Total: 20 marks)

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Revised Practice Paper 2

2025 historical bridge

Time allowed: 2 hours

Total marks: 80

Questions: Answer all four questions

Historical role: 2025 spectra, Thomson principle, random-walk hitting, spring modes and piecewise conduction

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

Question 1

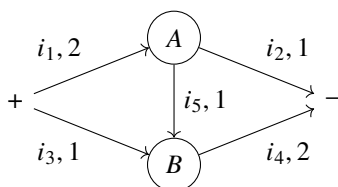
(a) Find the rank and eigenvalues of

$$T = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}. (4marks)$$

(b) Find the rank and eigenvalues of the circulant matrix

$$C = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{pmatrix}. (4marks)$$

(c) The five edges of the network below have conductances 2, 1, 1, 2, 1, respectively, in the order $+A$, $A-$, $+B$, $B-$, AB . A unit current enters at $+$ and leaves at $-$.



Parameterise the currents by

$$i_1 = x, \quad i_3 = 1 - x, \quad i_2 = x - y, \quad i_5 = y, \quad i_4 = 1 - x + y.$$

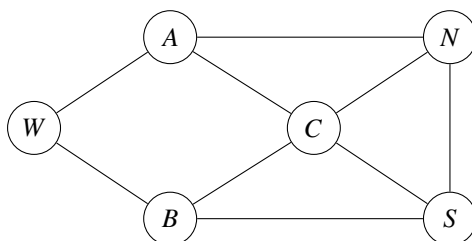
(i) Write the Thomson energy as a function of x, y . (4 marks)

(ii) Minimise it to find all five currents. (4 marks)

(iii) Find the effective conductance and verify it by solving for the node voltages. (4 marks) (Total: 20 marks)

Question 2

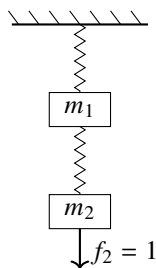
Consider the graph below. A random walker chooses uniformly among incident edges.



- (a) Give the dimensions of an incidence matrix (edges by vertices) and the dimensions of its right and left null spaces. (4 marks)
- (b) Starting at W , find the probability of reaching S before returning to W . (6 marks)
- (c) Starting at N , find the probability of hitting W before S . (6 marks)
- (d) Starting at C , find the probability of reaching N before either W or S . (4 marks) (Total: 20 marks)

Question 3

Two unit masses hang vertically below a fixed ceiling. A unit spring joins the ceiling to mass 1 and a second unit spring joins mass 1 to mass 2. Gravity is ignored. A constant force $f = (0, 1)^T$ acts on the masses.



- (a) Write the stiffness matrix and find the static equilibrium. (4 marks)
 - (b) Find the reaction force at the ceiling. (2 marks)
 - (c) Find the two natural frequencies and corresponding mode shapes. (6 marks)
 - (d) The force remains unchanged and the equilibrium is perturbed by $\hat{x}(0) = (1, 0)^T$, $\hat{x}'(0) = 0$. Find $\hat{x}(t)$ in modal form. (8 marks)
- (Total: 20 marks)

Question 4

A unit-conductivity rod occupies $0 \leq x \leq 2$. The current-source density is

$$F(x) = \begin{cases} -f, & 0 < x < 1, \\ 2f, & 1 < x < 2, \end{cases} \quad f > 0,$$

and $\phi(0) = 0$, $\phi(2) = V$. Recall $J = -\phi'$ and $J' = F$.

- (a) Write the differential equations in the two subintervals and state the interface conditions at $x = 1$. (3 marks)
- (b) Find the voltage $\phi(x)$ in both subintervals. (8 marks)
- (c) Find the relation between V and f that makes $J(0) = 0$. (3 marks)
- (d) Under this relation, find the location and value of the maximum voltage. (2 marks)
- (e) Sketch ϕ and J , showing their values at $x = 0, 1, 2$. (4 marks)

(Total: 20 marks)

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Revised Practice Paper 3

2023–2024 historical abilities

Time allowed: 2 hours

Total marks: 80

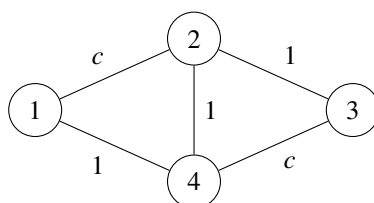
Questions: Answer all four questions

Historical role: Parameter circuits, tree recurrences, complete-graph spectra and multi-source complex conduction

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

Question 1

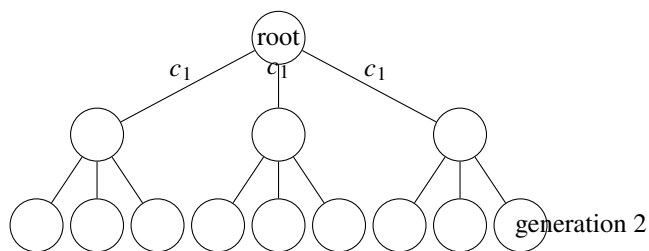
The circuit below has node 1 at unit voltage and node 3 grounded. Conductances are shown.



- (a) Write the conductance-weighted Laplacian in the node order 1, 3, 2, 4. (2 marks)
- (b) Find the effective conductance in the limits $c \rightarrow 0$ and $c \rightarrow \infty$. (2 marks)
- (c) Find the voltages at nodes 2 and 4 for general $c > 0$. (6 marks)
- (d) Find the effective conductance $C(c)$ and check both limits. (4 marks)
- (e) Put a copy with parameter c in series with a copy with parameter $1/c$. Find the effective conductance of the composite network. (6 marks) (Total: 20 marks)

Question 2

A ternary tree has n generations. The root is held at unit voltage, all generation- n leaves are grounded, and every edge between generations $j - 1$ and j has conductance $c_j > 0$.



- (a) Find the numbers of vertices and edges and derive a formula for the effective resistance in terms of the c_j . (4 marks)
- (b) When $c_j = 1$, find the effective conductance and the voltage at every node in generation k . Find the limit as $n \rightarrow \infty$. (8 marks)
- (c) Find the infinite-tree effective conductance when $c_j = 1/j$. (4 marks)
- (d) Find it when $c_j = j$. (4 marks)
- (Total: 20 marks)

Question 3

Consider the complete graph K_5 with unit edge weights.

- (a) Write its graph Laplacian. (2 marks)
- (b) Find the effective conductance between nodes 1 and 2. (4 marks)
- (c) Give a real orthonormal eigenbasis and all eigenvalues. (6 marks)
- (d) A unit current enters at node 1 and leaves at node 2. In the gauge $\sum_i x_i = 0$, find the voltage vector by eigenvector expansion. (6 marks)
- (e) Use the voltage difference to check part (b). (2 marks) (Total: 20 marks)

Question 4

The right half-plane $x > 0$ is a unit conductor. Let $0 < a < b$ and

$$h(z) = -\frac{m}{2\pi} \log \left(\frac{(z-a)(z+b)}{(z+a)(z-b)} \right), \quad \phi = \Re h(z).$$

- (a) Show that the wall $x = 0$ is grounded. (2 marks)
 - (b) Identify all sources and sinks lying in the conductor. (4 marks)
 - (c) Find $J_x - iJ_y = -h'(z)$. (3 marks)
 - (d) Show that $J_y = 0$ on the wall. (1 marks)
 - (e) Find $J_x(0, y)$ when $a = 1, b = 3$. (2 marks)
 - (f) Calculate the total signed current through the wall. (4 marks)
 - (g) Sketch $J_x(0, y)$, showing its zeros and the value at $y = 0$. (4 marks)
- (Total: 20 marks)

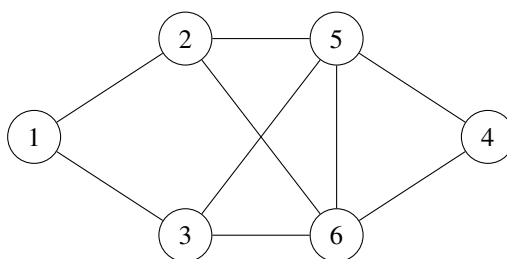
MATH40007 - An Introduction to Applied Mathematics
Revised Practice Paper 4
 2020–2022 historical abilities

Time allowed: 2 hours
 Total marks: 80
 Questions: Answer all four questions
 Historical role: Dirichlet networks, Green matrices, block spectra and image-electrode limits

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

Question 1

All edges of the six-node graph below have unit conductance.



- (a) Write the 6×6 graph Laplacian. (4 marks)
- (b) Set node 1 to unit voltage and node 4 to zero. Find all other voltages and the effective conductance. (8 marks)
- (c) Instead feed a unit current into node 1 and remove it at node 4. Find the terminal voltage difference and the current in edge 3–6. (8 marks) (Total: 20 marks)

Question 2

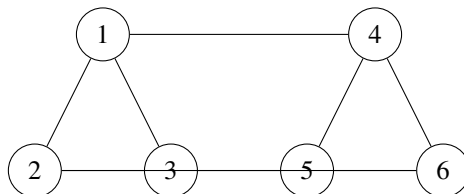
Consider the path graph $0 - 1 - 2 - 3 - 4$, with nodes 0 and 4 grounded and unit conductances. Let K be the reduced Laplacian on nodes 1,2,3.

- (a) For each $j = 1, 2, 3$, set node j to unit voltage while the other internal nodes satisfy KCL. Find the internal voltage vector. (6 marks)
- (b) Find the current divergence at the unit-voltage node in each case. (4 marks)
- (c) Prove the reciprocity identity $x^T \widehat{f} = \widehat{x}^T f$ for any two voltage/divergence pairs. (4 marks)
- (d) Find the matrix M such that $x = Mf$ for every divergence vector f . (6 marks) (Total: 20 marks)

Question 3

A triangular prism graph consists of two triangles with corresponding vertices joined. In block form its Laplacian is

$$K = \begin{pmatrix} A & -I_3 \\ -I_3 & A \end{pmatrix}, \quad A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$



- (a) Explain the block form of K . (4 marks)
- (b) Use vectors $(e, e)^T$ and $(e, -e)^T$ to find a complete orthonormal eigensystem. (8 marks)
- (c) For diffusion $x' = -Kx$, describe the long-time behaviour and identify the slowest non-constant decay rate. (8 marks)
- (Total: 20 marks)*

Question 4

The conductor is the region $x > 0$ outside a circular electrode. Define

$$h(z) = \frac{V}{\log \rho} \log \left(\frac{a-z}{a+z} \right), \quad 0 < \rho < 1,$$

and take the electrode boundary to be $\left| \frac{a-z}{a+z} \right| = \rho$.

- (a) Show that the wall $x = 0$ is grounded. (3 marks)
- (b) Show that the electrode is a circle, find its centre and radius, and show that it is at voltage V . (5 marks)
- (c) Find the complex current density. (4 marks)
- (d) Find the wall-current density, its maximum magnitude and the total outward current through the wall. (4 marks)
- (e) Show that the limit $\rho \rightarrow 0$, $V/\log \rho \rightarrow -m/(2\pi)$ gives a point source of strength m at $z = a$ near a grounded wall. (4 marks) (Total: 20 marks)

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Revised Practice Paper 5

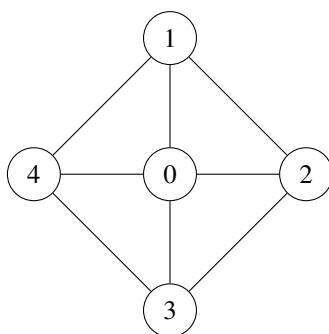
Seven-year gap-closing paper

Time allowed: 2 hours
 Total marks: 80
 Questions: Answer all four questions
 Historical role: Cycle spaces, non-standard hitting problems, generalised modes and slit-electrode singularity

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

Question 1

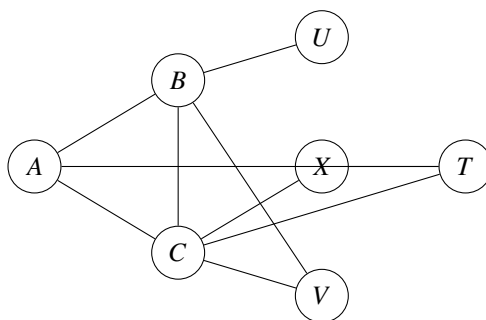
The wheel graph W_5 has a central node 0 joined to four outer nodes 1–4, and the outer nodes form a cycle.



- (a) Give the dimensions and ranks/nullities of an incidence matrix (edges by vertices). (4 marks)
- (b) Write the graph Laplacian and find the dimension of its null space. (4 marks)
- (c) Find a real orthonormal eigenbasis and all eigenvalues. (12 marks) (Total: 20 marks)

Question 2

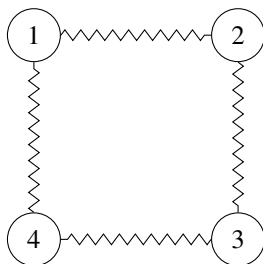
Consider the graph below. U and V are target stations, T is a stop station and all choices are uniform.



- (a) Give the dimensions and nullities of an incidence matrix. (4 marks)
- (b) Among starting nodes A, B, C , which maximises the probability of reaching U or V before T ? Give all three probabilities. (6 marks)
- (c) Starting at T , find the probability of reaching U or V before returning to T . (6 marks)
- (d) Close station V and remove its incident edges. Starting at A , find the probability of reaching C before returning to A . (4 marks)
- (Total: 20 marks)

Question 3

Four masses form a square cycle of unit springs. Masses 1 and 3 have mass 1; masses 2 and 4 have mass $1/4$.



Let K be the cycle Laplacian, $M = \text{diag}(1, 1/4, 1, 1/4)$ and $\hat{K} = M^{-1/2} K M^{-1/2}$.

- (a) Write K and $\hat{K}$. (4 marks)
- (b) Let S exchange nodes $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$. Explain why $S^2 = I$ and $S\hat{K} = \hat{K}S$. (4 marks)
- (c) Use the $S = +1$ and $S = -1$ subspaces to find all eigenvalues and eigenvectors of $\hat{K}$. (8 marks)
- (d) Give all natural frequencies and identify the rigid mode. (4 marks)

(Total: 20 marks)

Question 4

The upper half-plane is a unit conductor. The interval $-1 < x < 1$ on the boundary is a grounded electrode, while the rest of the boundary is insulated. Let

$$h(z) = -\frac{m}{2\pi} \log \left(z + \sqrt{z^2 - 1} \right),$$

where the branch is chosen so that $\sqrt{z^2 - 1} \sim z$ at infinity.

- (a) Show that the electrode is grounded. (4 marks)
- (b) Identify the source represented by the far-field behaviour. (4 marks)
- (c) Find the complex current density. (4 marks)
- (d) Find the normal current density on the electrode and its total flux. (4 marks)
- (e) Find the current on the insulated boundary $x > 1$ and its leading singular behaviour as $x \downarrow 1$. (4 marks) (*Total: 20 marks*)

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Revised Practice Paper 6

Seven-year mixed transfer pressure test

Time allowed: 2 hours
 Total marks: 80
 Questions: Answer all four questions
 Historical role: Topics reordered to prevent recognition by 2026 question position

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

Question 1

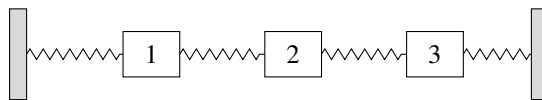
A unit conductor occupies the first quadrant. Let $z_0 = 1 + i$ and

$$h(z) = -\frac{m}{2\pi} \log \left(\frac{z^2 - z_0^2}{z^2 - \bar{z}_0^2} \right), \quad \phi = \Re h(z).$$

- (a) Show that both coordinate axes are grounded. (4 marks)
- (b) Identify the source inside the conductor. (4 marks)
- (c) Find the complex current density. (4 marks)
- (d) Find the outward normal-current density on each grounded axis. (4 marks)
- (e) Calculate the flux through each axis and verify current conservation. (4 marks) (Total: 20 marks)

Question 2

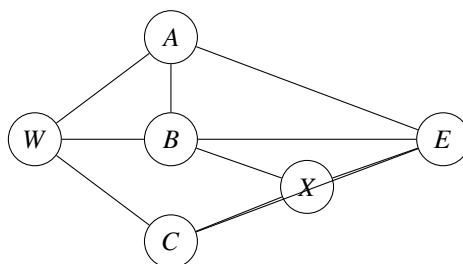
Three unit masses move between two fixed walls. Unit springs join the left wall to mass 1, successive masses, and mass 3 to the right wall.



- (a) Write the stiffness matrix and find an orthonormal eigensystem and the natural frequencies. (8 marks)
- (b) A constant force $f = (1, 0, 1)^T$ is applied. Find the equilibrium. (4 marks)
- (c) The force is removed at $t = 0$ with initial displacement equal to the equilibrium and initial velocity $(0, 1, 0)^T$. Find the free motion in modal form. (8 marks) *(Total: 20 marks)*

Question 3

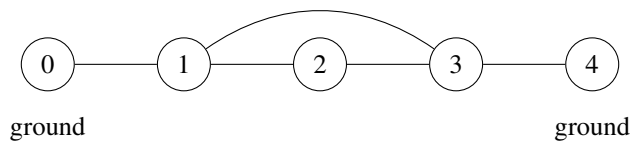
Consider the graph below, with terminals W and E .



- (a) Give the dimensions and nullities of an incidence matrix. (4 marks)
- (b) A walker starts at W , leaves it and stops on reaching E or returning to W . Find the probability of reaching E first. (8 marks)
- (c) If W is held at voltage 1 and E grounded, find the voltage at X . (8 marks) (Total: 20 marks)

Question 4

Nodes 0 and 4 are grounded. Internal nodes 1,2,3 are connected by the path $0 - 1 - 2 - 3 - 4$ and there is one extra unit edge $1 - 3$.



- (a) Write the reduced Laplacian K . (4 marks)
 - (b) Find the Green matrix $M = K^{-1}$. (8 marks)
 - (c) A unit current is injected at node 2 and removed through the grounds. Find the internal voltages and verify that the total ground current is one. (4 marks)
 - (d) A unit current is injected at node 1 and removed at node 3. Find the effective resistance and conductance between nodes 1 and 3. (4 marks)
- (Total: 20 marks)*