

MATH40007 - An Introduction to Applied Mathematics

Revised Practice Paper 2

2025 historical bridge

Time allowed: 2 hours

Total marks: 80

Questions: Answer all four questions

Historical role: 2025 spectra, Thomson principle, random-walk hitting, spring modes and piecewise conduction

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

Question 1

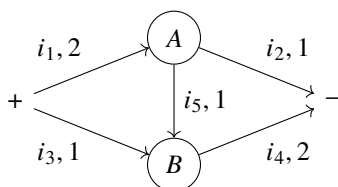
(a) Find the rank and eigenvalues of

$$T = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}. (4marks)$$

(b) Find the rank and eigenvalues of the circulant matrix

$$C = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{pmatrix}. (4marks)$$

(c) The five edges of the network below have conductances 2, 1, 1, 2, 1, respectively, in the order $+A$, $A-$, $+B$, $B-$, AB . A unit current enters at $+$ and leaves at $-$.



Parameterise the currents by

$$i_1 = x, \quad i_3 = 1 - x, \quad i_2 = x - y, \quad i_5 = y, \quad i_4 = 1 - x + y.$$

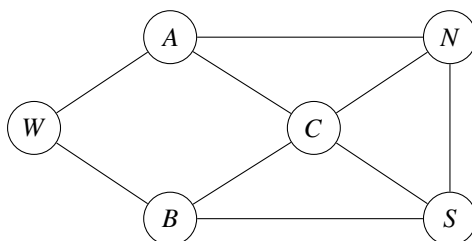
(i) Write the Thomson energy as a function of x, y . (4 marks)

(ii) Minimise it to find all five currents. (4 marks)

(iii) Find the effective conductance and verify it by solving for the node voltages. (4 marks) (Total: 20 marks)

Question 2

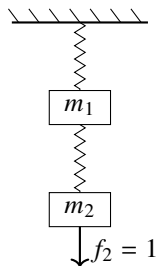
Consider the graph below. A random walker chooses uniformly among incident edges.



- (a) Give the dimensions of an incidence matrix (edges by vertices) and the dimensions of its right and left null spaces. (4 marks)
- (b) Starting at W , find the probability of reaching S before returning to W . (6 marks)
- (c) Starting at N , find the probability of hitting W before S . (6 marks)
- (d) Starting at C , find the probability of reaching N before either W or S . (4 marks) (Total: 20 marks)

Question 3

Two unit masses hang vertically below a fixed ceiling. A unit spring joins the ceiling to mass 1 and a second unit spring joins mass 1 to mass 2. Gravity is ignored. A constant force $f = (0, 1)^T$ acts on the masses.



- (a) Write the stiffness matrix and find the static equilibrium. (4 marks)
 - (b) Find the reaction force at the ceiling. (2 marks)
 - (c) Find the two natural frequencies and corresponding mode shapes. (6 marks)
 - (d) The force remains unchanged and the equilibrium is perturbed by $\hat{x}(0) = (1, 0)^T$, $\hat{x}'(0) = 0$. Find $\hat{x}(t)$ in modal form. (8 marks)
- (Total: 20 marks)

Question 4

A unit-conductivity rod occupies $0 \leq x \leq 2$. The current-source density is

$$F(x) = \begin{cases} -f, & 0 < x < 1, \\ 2f, & 1 < x < 2, \end{cases} \quad f > 0,$$

and $\phi(0) = 0$, $\phi(2) = V$. Recall $J = -\phi'$ and $J' = F$.

- (a) Write the differential equations in the two subintervals and state the interface conditions at $x = 1$. (3 marks)
- (b) Find the voltage $\phi(x)$ in both subintervals. (8 marks)
- (c) Find the relation between V and f that makes $J(0) = 0$. (3 marks)
- (d) Under this relation, find the location and value of the maximum voltage. (2 marks)
- (e) Sketch ϕ and J , showing their values at $x = 0, 1, 2$. (4 marks)

(Total: 20 marks)