

MATH40007 - An Introduction to Applied Mathematics

Revised Practice Paper 3

2023–2024 historical abilities

Time allowed: 2 hours

Total marks: 80

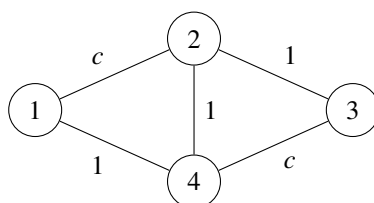
Questions: Answer all four questions

Historical role: Parameter circuits, tree recurrences, complete-graph spectra and multi-source complex conduction

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

Question 1

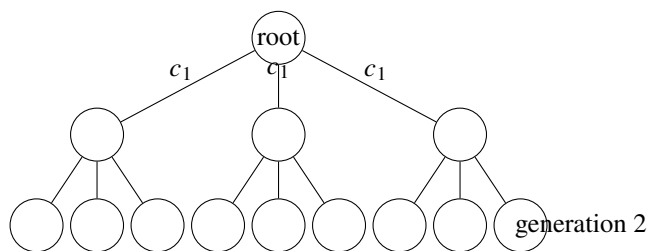
The circuit below has node 1 at unit voltage and node 3 grounded. Conductances are shown.



- (a) Write the conductance-weighted Laplacian in the node order 1, 3, 2, 4. (2 marks)
- (b) Find the effective conductance in the limits $c \rightarrow 0$ and $c \rightarrow \infty$. (2 marks)
- (c) Find the voltages at nodes 2 and 4 for general $c > 0$. (6 marks)
- (d) Find the effective conductance $C(c)$ and check both limits. (4 marks)
- (e) Put a copy with parameter c in series with a copy with parameter $1/c$. Find the effective conductance of the composite network. (6 marks) (Total: 20 marks)

Question 2

A ternary tree has n generations. The root is held at unit voltage, all generation- n leaves are grounded, and every edge between generations $j - 1$ and j has conductance $c_j > 0$.



- (a) Find the numbers of vertices and edges and derive a formula for the effective resistance in terms of the c_j . (4 marks)
- (b) When $c_j = 1$, find the effective conductance and the voltage at every node in generation k . Find the limit as $n \rightarrow \infty$. (8 marks)
- (c) Find the infinite-tree effective conductance when $c_j = 1/j$. (4 marks)
- (d) Find it when $c_j = j$. (4 marks)
- (Total: 20 marks)

Question 3

Consider the complete graph K_5 with unit edge weights.

- (a) Write its graph Laplacian. (2 marks)
- (b) Find the effective conductance between nodes 1 and 2. (4 marks)
- (c) Give a real orthonormal eigenbasis and all eigenvalues. (6 marks)
- (d) A unit current enters at node 1 and leaves at node 2. In the gauge $\sum_i x_i = 0$, find the voltage vector by eigenvector expansion. (6 marks)
- (e) Use the voltage difference to check part (b). (2 marks) *(Total: 20 marks)*

Question 4

The right half-plane $x > 0$ is a unit conductor. Let $0 < a < b$ and

$$h(z) = -\frac{m}{2\pi} \log\left(\frac{(z-a)(z+b)}{(z+a)(z-b)}\right), \quad \phi = \Re h(z).$$

- (a) Show that the wall $x = 0$ is grounded. (2 marks)
 - (b) Identify all sources and sinks lying in the conductor. (4 marks)
 - (c) Find $J_x - iJ_y = -h'(z)$. (3 marks)
 - (d) Show that $J_y = 0$ on the wall. (1 marks)
 - (e) Find $J_x(0, y)$ when $a = 1, b = 3$. (2 marks)
 - (f) Calculate the total signed current through the wall. (4 marks)
 - (g) Sketch $J_x(0, y)$, showing its zeros and the value at $y = 0$. (4 marks)
- (Total: 20 marks)