

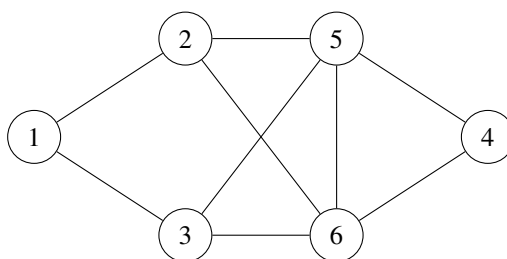
MATH40007 - An Introduction to Applied Mathematics
Revised Practice Paper 4
 2020–2022 historical abilities

Time allowed: 2 hours
 Total marks: 80
 Questions: Answer all four questions
 Historical role: Dirichlet networks, Green matrices, block spectra and image-electrode limits

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

Question 1

All edges of the six-node graph below have unit conductance.



- (a) Write the 6×6 graph Laplacian. (4 marks)
- (b) Set node 1 to unit voltage and node 4 to zero. Find all other voltages and the effective conductance. (8 marks)
- (c) Instead feed a unit current into node 1 and remove it at node 4. Find the terminal voltage difference and the current in edge 3–6. (8 marks) (Total: 20 marks)

Question 2

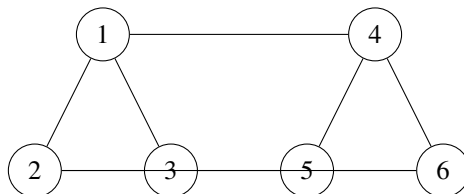
Consider the path graph $0 - 1 - 2 - 3 - 4$, with nodes 0 and 4 grounded and unit conductances. Let K be the reduced Laplacian on nodes 1,2,3.

- (a) For each $j = 1, 2, 3$, set node j to unit voltage while the other internal nodes satisfy KCL. Find the internal voltage vector. (6 marks)
- (b) Find the current divergence at the unit-voltage node in each case. (4 marks)
- (c) Prove the reciprocity identity $x^T \widehat{f} = \widehat{x}^T f$ for any two voltage/divergence pairs. (4 marks)
- (d) Find the matrix M such that $x = Mf$ for every divergence vector f . (6 marks) (Total: 20 marks)

Question 3

A triangular prism graph consists of two triangles with corresponding vertices joined. In block form its Laplacian is

$$K = \begin{pmatrix} A & -I_3 \\ -I_3 & A \end{pmatrix}, \quad A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$



- (a) Explain the block form of K . (4 marks)
- (b) Use vectors $(e, e)^T$ and $(e, -e)^T$ to find a complete orthonormal eigensystem. (8 marks)
- (c) For diffusion $x' = -Kx$, describe the long-time behaviour and identify the slowest non-constant decay rate. (8 marks)
- (Total: 20 marks)*

Question 4

The conductor is the region $x > 0$ outside a circular electrode. Define

$$h(z) = \frac{V}{\log \rho} \log \left(\frac{a-z}{a+z} \right), \quad 0 < \rho < 1,$$

and take the electrode boundary to be $\left| \frac{a-z}{a+z} \right| = \rho$.

- (a) Show that the wall $x = 0$ is grounded. (3 marks)
- (b) Show that the electrode is a circle, find its centre and radius, and show that it is at voltage V . (5 marks)
- (c) Find the complex current density. (4 marks)
- (d) Find the wall-current density, its maximum magnitude and the total outward current through the wall. (4 marks)
- (e) Show that the limit $\rho \rightarrow 0$, $V/\log \rho \rightarrow -m/(2\pi)$ gives a point source of strength m at $z = a$ near a grounded wall. (4 marks) (Total: 20 marks)