

MATH40007 - An Introduction to Applied Mathematics

Revised Practice Paper 5

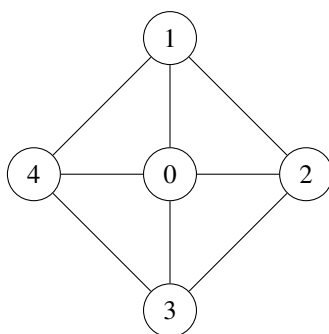
Seven-year gap-closing paper

Time allowed: 2 hours
 Total marks: 80
 Questions: Answer all four questions
 Historical role: Cycle spaces, non-standard hitting problems, generalised modes and slit-electrode singularity

Instructions. Show all essential working. Use the methods taught in the course. Credit may be awarded for a correct method even when a later arithmetic error occurs. This is unofficial practice material, not an Imperial College examination paper.

Question 1

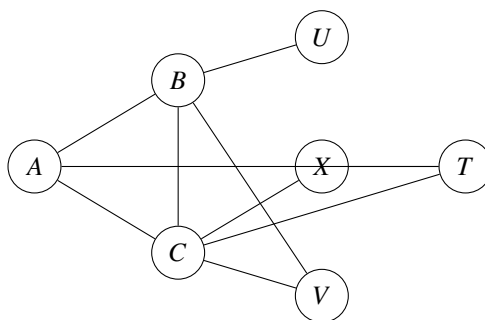
The wheel graph W_5 has a central node 0 joined to four outer nodes 1–4, and the outer nodes form a cycle.



- (a) Give the dimensions and ranks/nullities of an incidence matrix (edges by vertices). (4 marks)
- (b) Write the graph Laplacian and find the dimension of its null space. (4 marks)
- (c) Find a real orthonormal eigenbasis and all eigenvalues. (12 marks) (Total: 20 marks)

Question 2

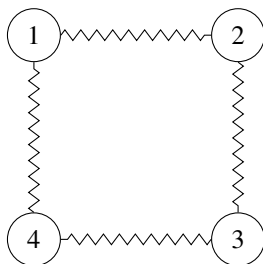
Consider the graph below. U and V are target stations, T is a stop station and all choices are uniform.



- (a) Give the dimensions and nullities of an incidence matrix. (4 marks)
- (b) Among starting nodes A, B, C , which maximises the probability of reaching U or V before T ? Give all three probabilities. (6 marks)
- (c) Starting at T , find the probability of reaching U or V before returning to T . (6 marks)
- (d) Close station V and remove its incident edges. Starting at A , find the probability of reaching C before returning to A . (4 marks)
- (Total: 20 marks)

Question 3

Four masses form a square cycle of unit springs. Masses 1 and 3 have mass 1; masses 2 and 4 have mass $1/4$.



Let K be the cycle Laplacian, $M = \text{diag}(1, 1/4, 1, 1/4)$ and $\widehat{K} = M^{-1/2} K M^{-1/2}$.

- (a) Write K and $\widehat{K}$. (4 marks)
- (b) Let S exchange nodes $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$. Explain why $S^2 = I$ and $S\widehat{K} = \widehat{K}S$. (4 marks)
- (c) Use the $S = +1$ and $S = -1$ subspaces to find all eigenvalues and eigenvectors of $\widehat{K}$. (8 marks)
- (d) Give all natural frequencies and identify the rigid mode. (4 marks)

(Total: 20 marks)

Question 4

The upper half-plane is a unit conductor. The interval $-1 < x < 1$ on the boundary is a grounded electrode, while the rest of the boundary is insulated. Let

$$h(z) = -\frac{m}{2\pi} \log \left(z + \sqrt{z^2 - 1} \right),$$

where the branch is chosen so that $\sqrt{z^2 - 1} \sim z$ at infinity.

- (a) Show that the electrode is grounded. (4 marks)
- (b) Identify the source represented by the far-field behaviour. (4 marks)
- (c) Find the complex current density. (4 marks)
- (d) Find the normal current density on the electrode and its total flux. (4 marks)
- (e) Find the current on the insulated boundary $x > 1$ and its leading singular behaviour as $x \downarrow 1$. (4 marks) (*Total: 20 marks*)