

Paper 1 - Solutions and Marking Guide

2026 structure reinforcement

Question 1

Recognition cue. The columns are visibly dependent, so determine one pivot column first and then use rank-nullity.

(a) The columns are $c_2 = 2c_1$ and $c_3 = c_1$, so $\text{rank } M = 1$.

$$\text{Col } M = \text{span}\{(1, 2, 3, 4)^T\}, \quad \text{Row } M = \text{span}\{(1, 2, 1)\}.$$

The equation $x_1 + 2x_2 + x_3 = 0$ gives

$$\mathcal{N}(M) = \text{span}\{(-2, 1, 0)^T, (-1, 0, 1)^T\}.$$

The left-null condition $y_1 + 2y_2 + 3y_3 + 4y_4 = 0$ gives, for example,

$$\mathcal{N}(M^T) = \text{span}\{(-2, 1, 0, 0)^T, (-3, 0, 1, 0)^T, (-4, 0, 0, 1)^T\}.$$

(b) Put $v_3 = a$, $v_4 = b$. KCL gives

$$2(a - 1) + a + (a - b) = 0, \quad (b - 1) + 2b + (b - a) = 0.$$

Hence $a = 3/5$, $b = 2/5$. The edge current is $a - b = 1/5$, from 3 to 4. The input current is

$$2(1 - a) + (1 - b) = \frac{4}{5} + \frac{3}{5} = \frac{7}{5}.$$

so $C_{\text{eff}} = 7/5$. **Checks.** Potentials lie between 0 and 1; current flows from the higher-potential node; input and output currents balance.

Question difficulty allocation: $A = 8$, $B = 12$, $C = 0$, $D = 0$.

Question 2

Recognition cue. Solve harmonic equations for the bridge, then use series/parallel resistance and the escape-probability identity $p = C_{\text{eff}}/\deg(+)$.

(a) Let the potentials at A, B be a, b . Then

$$4a - b = 1, \quad -a + 3b = 1,$$

so $a = 4/11$, $b = 5/11$. The input current is

$$(1 - a) + (1 - b) = \frac{13}{11}.$$

(b) The two path conductances are $1/2$ and $1/4$, hence $C_B = 3/4$.

(c) The total resistance is

$$R = \frac{11}{13} + 1 + \frac{4}{3} = \frac{124}{39}, \quad C = \frac{39}{124}.$$

The initial terminal has degree 2, so

$$p = \frac{C}{2} = \frac{39}{248}.$$

Checks. All equivalent conductances are positive and smaller than the total number of edges leaving the source.

Question difficulty allocation: $A = 6$, $B = 0$, $C = 8$, $D = 6$.

Question 3

Recognition cue. For a forced diffusion system, first find the steady state, then expand the initial deviation rather than the steady state.

The reduced equation is

$$\hat{x}' = -K_2 \hat{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

The steady state is $x_s = (4/3, 2/3)^T$. An orthonormal eigensystem is

$$\phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_1 = 1; \quad \phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = 3.$$

Since $\hat{x}(0) - x_s = (-4/3, 1/3)^T$, the modal coefficients are

$$\alpha_1 = -\frac{1}{\sqrt{2}}, \quad \alpha_2 = -\frac{5}{3\sqrt{2}}.$$

Thus

$$x_2(t) = \frac{4}{3} - \frac{1}{2}e^{-t} - \frac{5}{6}e^{-3t}, \quad x_3(t) = \frac{2}{3} - \frac{1}{2}e^{-t} + \frac{5}{6}e^{-3t}.$$

The initial values are 0, 1 and the limits are $4/3, 2/3$. The first component rises monotonically; the second initially falls and later rises to its limit. **Checks.** Substitute $t = 0$ and $t \rightarrow \infty$ immediately. The exponents must be the eigenvalues 1 and 3.

Question difficulty allocation: $A = 8, B = 8, C = 4, D = 0$.

Question 4

Recognition cue. Use current conservation before solving the heat equation.

No source implies $J' = 0$, so $\phi'' = 0$. Therefore

$$\phi = x, \quad J = -2, \quad q_e = -\phi' J = 2.$$

The heat equation is $-3T'' = 2$. With $T(0) = 0, T(1) = 1$,

$$T = -\frac{x^2}{3} + \frac{4x}{3}.$$

For insulation at $x = 0, T'(0) = 0$ and $T(1) = 1$, so

$$T = \frac{4}{3} - \frac{x^2}{3}.$$

The outward flux at $x = 1$ is $-3T'(1) = 2$. This equals $\int_0^1 q_e dx = 2$. **Question difficulty allocation:** $A = 10, B = 0, C = 0, D = 10$.

Paper 2 - Solutions and Marking Guide

2025 historical bridge

Question 1

Recognition cue. Use the standard tridiagonal formula, the Fourier eigenvectors of a circulant, and direct differentiation of Thomson energy.

(a) For the 3×3 tridiagonal matrix,

$$\lambda_k = 3 - 2 \cos \frac{k\pi}{4}, \quad k = 1, 2, 3,$$

so the eigenvalues are $3 - \sqrt{2}$, 3 , $3 + \sqrt{2}$ and the rank is 3.

(b) With fourth roots of unity, the circulant eigenvalues are

$$1 + 2\omega^k + 2\omega^{3k}, \quad k = 0, 1, 2, 3,$$

namely 5, 1, -3 , 1. Hence the rank is 4.

(c) The energy is

$$E(x, y) = \frac{x^2}{2} + (x - y)^2 + (1 - x)^2 + \frac{(1 - x + y)^2}{2} + y^2.$$

The stationarity equations are $2x - y = 1$ and $-3x + 5y = -1$, giving

$$x = \frac{4}{7}, \quad y = \frac{1}{7}.$$

Thus

$$(i_1, i_2, i_3, i_4, i_5) = \left(\frac{4}{7}, \frac{3}{7}, \frac{3}{7}, \frac{4}{7}, \frac{1}{7} \right).$$

The minimum energy is $5/7$, so $R_{\text{eff}} = 5/7$ and

$$C_{\text{eff}} = \frac{7}{5}.$$

A voltage calculation gives $V_A = 3/5$, $V_B = 2/5$ under unit terminal voltage, and input current $7/5$, confirming the result.

Question difficulty allocation: $A = 4$, $B = 8$, $C = 4$, $D = 4$.

Question 2

Recognition cue. Set boundary values first and solve the small harmonic system. Do not try to enumerate paths.

There are 9 edges and 6 vertices, so an edge-by-vertex incidence matrix is 9×6 . The graph is connected: right-null dimension 1, rank 5 and left-null dimension $9 - 5 = 4$.

For $P(S \text{ before } W)$, the harmonic values are

$$P_A = \frac{15}{32}, \quad P_B = \frac{9}{16}, \quad P_C = \frac{11}{16}, \quad P_N = \frac{23}{32}.$$

After leaving W , the walker chooses A or B , hence

$$P_W = \frac{1}{2}(P_A + P_B) = \frac{33}{64}.$$

For $P(W \text{ before } S)$ the corresponding values are

$$P_A = \frac{17}{32}, \quad P_B = \frac{7}{16}, \quad P_C = \frac{5}{16}, \quad P_N = \frac{9}{32}.$$

For $P(N \text{ before } \{W, S\})$ one obtains $P_A = 7/15$, $P_B = 2/15$ and

$$P_C = \frac{2}{5}.$$

Question difficulty allocation: $A = 4$, $B = 6$, $C = 6$, $D = 4$.

Question 3

The stiffness matrix and force balance are

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad Kx_e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence $x_e = (1, 2)^T$. The ceiling spring is stretched by 1, so the reaction has magnitude 1 in the opposite direction.

The characteristic equation is $\lambda^2 - 3\lambda + 1 = 0$, so

$$\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}, \quad \omega_{\pm} = \sqrt{\lambda_{\pm}},$$

with mode vectors $r_{\pm} = (1, 2 - \lambda_{\pm})^T$. For the perturbation $\widehat{x}(0) = e_1, \widehat{x}'(0) = 0$,

$$\widehat{x}(t) = \sum_{\sigma=\pm} \frac{\cos(\omega_{\sigma}t)}{1 + (2 - \lambda_{\sigma})^2} r_{\sigma}.$$

Checks. At $t = 0$ the two modal projections must sum to $(1, 0)^T$.

Question difficulty allocation: $A = 8, B = 6, C = 2, D = 4$.

Question 4

Since $J = -\phi'$ and $J' = F$,

$$\phi''_- = f, \quad 0 < x < 1; \quad \phi''_+ = -2f, \quad 1 < x < 2.$$

Both ϕ and $J = -\phi'$ are continuous at $x = 1$. Solving with the endpoint conditions gives

$$\phi_-(x) = \frac{f}{2}x^2 + \left(\frac{V}{2} - \frac{f}{4}\right)x,$$

$$\phi_+(x) = -fx^2 + \left(\frac{V}{2} + \frac{11f}{4}\right)x - \frac{3f}{2}.$$

Now $J(0) = -V/2 + f/4$, so $J(0) = 0$ iff $V = f/2$. Then

$$\phi_- = \frac{f}{2}x^2, \quad \phi_+ = -fx^2 + 3fx - \frac{3f}{2}.$$

The maximum occurs where $\phi'_+ = -2fx + 3f = 0$, namely $x = 3/2$, and its value is $3f/4$. Also

$$J(0) = 0, \quad J(1) = -f, \quad J(2) = f.$$

These values determine the required sketches. **Question difficulty allocation:** $A = 16, B = 0, C = 0, D = 4$.

Paper 3 - Solutions and Marking Guide

2023–2024 historical abilities

Question 1

In the order 1, 3, 2, 4 the weighted Laplacian is

$$L = \begin{pmatrix} c+1 & 0 & -c & -1 \\ 0 & c+1 & -1 & -c \\ -c & -1 & c+2 & -1 \\ -1 & -c & -1 & c+2 \end{pmatrix}.$$

The limiting networks give $C(0) = 1/3$ and $C(\infty) = 3$. KCL gives

$$v_2 = \frac{c+1}{c+3}, \quad v_4 = \frac{2}{c+3}.$$

The input current is

$$C(c) = c(1 - v_2) + (1 - v_4) = \frac{3c+1}{c+3}.$$

For a series connection with parameters c and $1/c$,

$$C_{\text{series}} = \frac{(c+3)(3c+1)}{2(5c^2+6c+5)}.$$

Question difficulty allocation: $A = 8$, $B = 4$, $C = 4$, $D = 4$.

Question 2

The vertex and edge counts are

$$V_n = 1 + 3 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}, \quad E_n = 3 + \cdots + 3^n = \frac{3^{n+1} - 3}{2}.$$

Each generation- j layer consists of 3^j equal conductors in parallel, so its resistance is $1/(3^j c_j)$. Hence

$$R_n = \sum_{j=1}^n \frac{1}{3^j c_j}, \quad C_n = R_n^{-1}.$$

For $c_j = 1$,

$$C_n = \frac{2}{1 - 3^{-n}}, \quad C_\infty = 2,$$

and the voltage in generation k is

$$v_k = \frac{3^{-k} - 3^{-n}}{1 - 3^{-n}}.$$

For $c_j = 1/j$,

$$R_\infty = \sum_{j \geq 1} \frac{j}{3^j} = \frac{3}{4}, \quad C_\infty = \frac{4}{3}.$$

For $c_j = j$,

$$R_\infty = \sum_{j \geq 1} \frac{1}{j 3^j} = \log \frac{3}{2}, \quad C_\infty = \frac{1}{\log(3/2)}.$$

Question difficulty allocation: $A = 4$, $B = 6$, $C = 4$, $D = 6$.

Question 3

For K_5 , $L = 5I - J$. The effective resistance between any two nodes is $2/5$, hence $C_{12} = 5/2$. A convenient orthonormal eigenbasis is

$$\begin{aligned} u_0 &= \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1)^T, \quad \lambda_0 = 0, \\ u_1 &= \frac{1}{\sqrt{2}}(1, -1, 0, 0, 0)^T, \quad u_2 = \frac{1}{\sqrt{6}}(1, 1, -2, 0, 0)^T, \\ u_3 &= \frac{1}{\sqrt{12}}(1, 1, 1, -3, 0)^T, \quad u_4 = \frac{1}{\sqrt{20}}(1, 1, 1, 1, -4)^T, \end{aligned}$$

all with eigenvalue 5 except u_0 . For $f = e_1 - e_2$, only the u_1 mode is present, so

$$x = \frac{1}{5}f = \left(\frac{1}{5}, -\frac{1}{5}, 0, 0, 0\right)^T.$$

The voltage difference is $2/5$, confirming $R = 2/5$ and $C = 5/2$. **Question difficulty allocation:** $A = 8, B = 4, C = 4, D = 4$.

Question 4

On $z = iy$, numerator and denominator in the logarithm have equal modulus, so $\phi = 0$. The finite singularities in $x > 0$ are a source $+m$ at $z = a$ and a sink $-m$ at $z = b$.

$$J_x - iJ_y = \frac{m}{2\pi} \left(\frac{1}{z-a} - \frac{1}{z+a} - \frac{1}{z-b} + \frac{1}{z+b} \right).$$

For $a = 1, b = 3$ and $z = iy$,

$$\boxed{J_x(0, y) = \frac{m}{\pi} \left(-\frac{1}{y^2 + 1} + \frac{3}{y^2 + 9} \right)}, \quad J_y = 0.$$

Its zeros are $y = \pm\sqrt{3}$, $J_x(0, 0) = -2m/(3\pi)$, and the tails are positive. The signed wall flux is zero because the source and sink strengths cancel:

$$\int_{-\infty}^{\infty} J_x(0, y) dy = 0.$$

Question difficulty allocation: $A = 12, B = 6, C = 0, D = 2$.

Paper 4 - Solutions and Marking Guide

2020–2022 historical abilities

Question 1

The Laplacian is

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & -1 & -1 \\ -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

With $V_1 = 1, V_4 = 0$, symmetry and KCL give

$$V_2 = V_3 = \frac{3}{5}, \quad V_5 = V_6 = \frac{2}{5},$$

and $C_{\text{eff}} = 2(1 - 3/5) = 4/5$. For unit current, scale all voltages by $1/(4/5) = 5/4$. Thus the terminal voltage difference is $5/4$,

$$V_2 = V_3 = \frac{3}{4}, \quad V_5 = V_6 = \frac{1}{2},$$

and the current from 3 to 6 is $3/4 - 1/2 = 1/4$. **Question difficulty allocation:** $A = 8, B = 4, C = 4, D = 4$.

Question 2

The reduced Laplacian is

$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The three unit-voltage responses are

$$(1, 2/3, 1/3)^T, \quad (1/2, 1, 1/2)^T, \quad (1/3, 2/3, 1)^T.$$

The corresponding source divergences are $4/3, 1, 4/3$. Reciprocity follows from $f = Kx, \hat{f} = K\hat{x}$ and symmetry:

$$x^T \hat{f} = x^T K \hat{x} = (Kx)^T \hat{x} = \hat{x}^T f.$$

The Green matrix is

$$M = K^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Question difficulty allocation: $A = 8, B = 4, C = 4, D = 4$.

Question 3

Let L_3 be the triangle Laplacian. Since $A = L_3 + I$, symmetric vectors $(e, e)^T$ have eigenvalues of L_3 , while antisymmetric vectors $(e, -e)^T$ have eigenvalues of $L_3 + 2I$. The triangle eigenvalues are $0, 3, 3$, hence the prism eigenvalues are

$$0, 3, 3, 2, 5, 5.$$

Use any orthonormal triangle basis $e_0 = (1, 1, 1)/\sqrt{3}$ and two orthonormal vectors perpendicular to it, then normalise (e, e) and $(e, -e)$ by $1/\sqrt{2}$. For $x' = -Kx$, the constant mode is conserved. Every other component decays exponentially. The slowest non-constant rate is

$$2.$$

If the initial mean is zero, $x(t) \rightarrow 0$. **Question difficulty allocation:** $A = 8, B = 8, C = 0, D = 4$.

Question 4

For $z = iy$, the ratio $(a - z)/(a + z)$ has modulus 1, hence the wall is grounded. The circle equation is

$$(x - x_0)^2 + y^2 = r^2, \quad x_0 = \frac{a(1 + \rho^2)}{1 - \rho^2}, \quad r = \frac{2a\rho}{1 - \rho^2}.$$

On it, $\Re \log((a - z)/(a + z)) = \log \rho$, so $\phi = V$.

$$J_x - iJ_y = -h'(z) = \frac{2aV}{\log \rho (a^2 - z^2)}.$$

On the wall,

$$J_x(0, y) = \frac{2aV}{\log \rho (a^2 + y^2)}, \quad J_y = 0.$$

Its magnitude is largest at $y = 0$. The total outward current is

$$-\int_{-\infty}^{\infty} J_x dy = \boxed{-\frac{2\pi V}{\log \rho}} > 0.$$

If $V/\log \rho \rightarrow -m/(2\pi)$ and $\rho \rightarrow 0$, the circle shrinks to $z = a$ and the logarithmic singularity has source strength m . **Question**

difficulty allocation: $A = 8$, $B = 4$, $C = 4$, $D = 4$.

Paper 5 - Solutions and Marking Guide

Seven-year gap-closing paper

Question 1

There are 8 edges and 5 vertices. An edge-by-vertex incidence matrix is 8×5 , rank 4, right-null dimension 1 and left-null dimension 4. The Laplacian in node order 0, 1, 2, 3, 4 is

$$L = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{pmatrix}.$$

An orthogonal eigenbasis before normalisation is

$$(1, 1, 1, 1, 1), \quad (0, 1, 0, -1, 0), \quad (0, 0, 1, 0, -1), \\ (0, 1, -1, 1, -1), \quad (-4, 1, 1, 1, 1),$$

with eigenvalues 0, 3, 3, 5, 5. Normalise by $\sqrt{5}, \sqrt{2}, \sqrt{2}, 2, \sqrt{20}$ respectively. **Question difficulty allocation:** $A = 8, B = 4, C = 4, D = 4$.

Question 2

There are 10 edges and 7 vertices, so the incidence matrix is 10×7 , rank 6, right-null dimension 1 and left-null dimension 4. For hitting U or V before T , harmonic equations give

$$P_A = \frac{32}{81}, \quad P_B = \frac{58}{81}, \quad P_C = \frac{38}{81}, \quad P_X = \frac{19}{81}.$$

Thus B is best. Starting at T and conditioning on the first step,

$$P_T = \frac{P_A + P_C + P_X}{3} = \frac{89}{243}.$$

After closing V , solve with boundary values $P_A = 0, P_C = 1$. The internal values are $P_B = P_U = 1/2, P_T = 3/5, P_X = 4/5$. After leaving A ,

$$P(\text{hit } C \text{ before return } A) = \frac{1/2 + 1 + 3/5}{3} = \frac{7}{10}.$$

Question difficulty allocation: $A = 4, B = 6, C = 6, D = 4$.

Question 3

The cycle Laplacian is

$$K = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix},$$

and

$$\widehat{K} = \begin{pmatrix} 2 & -2 & 0 & -2 \\ -2 & 8 & -2 & 0 \\ 0 & -2 & 2 & -2 \\ -2 & 0 & -2 & 8 \end{pmatrix}.$$

The half-turn is an involution and preserves all masses and springs, so it commutes with $\widehat{K}$. The eigenpairs are

$$0 : (2, 1, 2, 1), \quad 2 : (-1, 0, 1, 0), \quad 8 : (0, -1, 0, 1), \quad 10 : (-1, 2, -1, 2).$$

Therefore the natural frequencies are

$$0, \sqrt{2}, 2\sqrt{2}, \sqrt{10}.$$

The zero-frequency vector is the rigid translation in mass-scaled coordinates. **Question difficulty allocation:** $A = 8, B = 6, C = 2, D = 4$.

Question 4

For $|x| < 1$ approached from above,

$$\sqrt{x^2 - 1} = i\sqrt{1 - x^2},$$

and $|x + i\sqrt{1 - x^2}| = 1$, so $\phi = 0$. At infinity

$$h(z) \sim -\frac{m}{2\pi} \log(2z),$$

which supplies the net far-field current associated with strength m . Since

$$\frac{d}{dz} \log(z + \sqrt{z^2 - 1}) = \frac{1}{\sqrt{z^2 - 1}},$$

$$\boxed{J_x - iJ_y = \frac{m}{2\pi\sqrt{z^2 - 1}}}.$$

On the electrode,

$$J_x = 0, \quad J_y = \frac{m}{2\pi\sqrt{1 - x^2}},$$

so the inward flux into the conductor is

$$\int_{-1}^1 J_y dx = \frac{m}{2}.$$

For $x > 1$, $J_y = 0$ and

$$J_x = \frac{m}{2\pi\sqrt{x^2 - 1}} \sim \frac{m}{2\pi\sqrt{2(x - 1)}}.$$

Question difficulty allocation: $A = 12$, $B = 4$, $C = 0$, $D = 4$.

Paper 6 - Solutions and Marking Guide

Seven-year mixed transfer pressure test

Question 1

On either coordinate axis, z^2 is real and the numerator and denominator are complex conjugates, so their ratio has modulus 1 and $\phi = 0$. Inside the first quadrant the only finite logarithmic singularity is at $z_0 = 1 + i$, a source of strength m . Differentiation gives

$$J_x - iJ_y = \frac{mz}{\pi} \left(\frac{1}{z^2 - z_0^2} - \frac{1}{z^2 - \bar{z}_0^2} \right).$$

On the positive x -axis,

$$J_x = 0, \quad J_y = -\frac{4mx}{\pi(x^4 + 4)},$$

so the outward density is $4mx/[\pi(x^4 + 4)]$. On the positive y -axis,

$$J_x = -\frac{4my}{\pi(y^4 + 4)}, \quad J_y = 0,$$

and the outward density is $4my/[\pi(y^4 + 4)]$. Each flux is

$$\int_0^\infty \frac{4ms}{\pi(s^4 + 4)} ds = \frac{m}{2},$$

so the total outward flux is m , matching the source strength. **Question difficulty allocation:** $A = 8, B = 4, C = 4, D = 4$.

Question 2

The stiffness matrix is

$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

An orthonormal eigensystem is

$$\phi_1 = (1/2, 1/\sqrt{2}, 1/2)^T, \quad \lambda_1 = 2 - \sqrt{2},$$

$$\phi_2 = (1/\sqrt{2}, 0, -1/\sqrt{2})^T, \quad \lambda_2 = 2,$$

$$\phi_3 = (1/2, -1/\sqrt{2}, 1/2)^T, \quad \lambda_3 = 2 + \sqrt{2}.$$

The frequencies are $\omega_j = \sqrt{\lambda_j}$. Since $K(1, 1, 1)^T = (1, 0, 1)^T$, the equilibrium is $x_e = (1, 1, 1)^T$. Let $a_j = \phi_j^T x_e$ and $b_j = \phi_j^T(0, 1, 0)^T$. Then

$$a_1 = 1 + \frac{1}{\sqrt{2}}, \quad a_2 = 0, \quad a_3 = 1 - \frac{1}{\sqrt{2}}; \quad b_1 = \frac{1}{\sqrt{2}}, \quad b_2 = 0, \quad b_3 = -\frac{1}{\sqrt{2}}.$$

Therefore

$$x(t) = \sum_{j=1}^3 \left(a_j \cos \omega_j t + \frac{b_j}{\omega_j} \sin \omega_j t \right) \phi_j.$$

Question difficulty allocation: $A = 8, B = 6, C = 2, D = 4$.

Question 3

There are 10 edges and 6 vertices. The incidence matrix is 10×6 , rank 5, right-null dimension 1 and left-null dimension 5. With boundary values $E = 1, W = 0$, harmonic equations give

$$V_A = \frac{41}{79}, \quad V_B = \frac{44}{79}, \quad V_C = \frac{45}{79}, \quad V_X = \frac{56}{79}.$$

After leaving W , the walker chooses A, B, C equally, hence

$$P(E \text{ before return } W) = \frac{41 + 44 + 45}{3 \cdot 79} = \frac{130}{237}.$$

With $W = 1, E = 0$, the voltage is the complement, so

$$V_X = 1 - \frac{56}{79} = \frac{23}{79}.$$

Question difficulty allocation: $A = 4, B = 6, C = 6, D = 4$.

Question 4

The reduced Laplacian is

$$K = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

Direct inversion gives

$$M = K^{-1} = \begin{pmatrix} 5/8 & 1/2 & 3/8 \\ 1/2 & 1 & 1/2 \\ 3/8 & 1/2 & 5/8 \end{pmatrix}.$$

For $f = e_2$,

$$x = M e_2 = (1/2, 1, 1/2)^T.$$

The ground currents through $0 - 1$ and $3 - 4$ are $1/2$ and $1/2$, totalling one. For $f = e_1 - e_3$,

$$x = M f = (1/4, 0, -1/4)^T.$$

The voltage difference is $1/2$, so

$$R_{13} = \frac{1}{2}, \quad C_{13} = 2.$$

Question difficulty allocation: $A = 12, B = 4, C = 0, D = 4$.

Pack-level difficulty audit

Every paper has 80 marks and the same aggregate official-style distribution as the 2026 paper:

$$A = 32, \quad B = 20, \quad C = 12, \quad D = 16.$$

The content and question order vary across historical ability families; the equality is at paper-level difficulty, not at fixed 2026 topic slots.