

Paper 2 - Solutions and Marking Guide

2025 historical bridge

Question 1

Recognition cue. Use the standard tridiagonal formula, the Fourier eigenvectors of a circulant, and direct differentiation of Thomson energy.

(a) For the 3×3 tridiagonal matrix,

$$\lambda_k = 3 - 2 \cos \frac{k\pi}{4}, \quad k = 1, 2, 3,$$

so the eigenvalues are $3 - \sqrt{2}$, 3 , $3 + \sqrt{2}$ and the rank is 3.

(b) With fourth roots of unity, the circulant eigenvalues are

$$1 + 2\omega^k + 2\omega^{3k}, \quad k = 0, 1, 2, 3,$$

namely 5, 1, -3 , 1. Hence the rank is 4.

(c) The energy is

$$E(x, y) = \frac{x^2}{2} + (x - y)^2 + (1 - x)^2 + \frac{(1 - x + y)^2}{2} + y^2.$$

The stationarity equations are $2x - y = 1$ and $-3x + 5y = -1$, giving

$$x = \frac{4}{7}, \quad y = \frac{1}{7}.$$

Thus

$$(i_1, i_2, i_3, i_4, i_5) = \left(\frac{4}{7}, \frac{3}{7}, \frac{3}{7}, \frac{4}{7}, \frac{1}{7} \right).$$

The minimum energy is $5/7$, so $R_{\text{eff}} = 5/7$ and

$$C_{\text{eff}} = \frac{7}{5}.$$

A voltage calculation gives $V_A = 3/5$, $V_B = 2/5$ under unit terminal voltage, and input current $7/5$, confirming the result.

Question difficulty allocation: $A = 4$, $B = 8$, $C = 4$, $D = 4$.

Question 2

Recognition cue. Set boundary values first and solve the small harmonic system. Do not try to enumerate paths.

There are 9 edges and 6 vertices, so an edge-by-vertex incidence matrix is 9×6 . The graph is connected: right-null dimension 1, rank 5 and left-null dimension $9 - 5 = 4$.

For $P(S \text{ before } W)$, the harmonic values are

$$P_A = \frac{15}{32}, \quad P_B = \frac{9}{16}, \quad P_C = \frac{11}{16}, \quad P_N = \frac{23}{32}.$$

After leaving W , the walker chooses A or B , hence

$$P_W = \frac{1}{2}(P_A + P_B) = \frac{33}{64}.$$

For $P(W \text{ before } S)$ the corresponding values are

$$P_A = \frac{17}{32}, \quad P_B = \frac{7}{16}, \quad P_C = \frac{5}{16}, \quad P_N = \frac{9}{32}.$$

For $P(N \text{ before } \{W, S\})$ one obtains $P_A = 7/15$, $P_B = 2/15$ and

$$P_C = \frac{2}{5}.$$

Question difficulty allocation: $A = 4$, $B = 6$, $C = 6$, $D = 4$.

Question 3

The stiffness matrix and force balance are

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad Kx_e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence $x_e = (1, 2)^T$. The ceiling spring is stretched by 1, so the reaction has magnitude 1 in the opposite direction.

The characteristic equation is $\lambda^2 - 3\lambda + 1 = 0$, so

$$\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}, \quad \omega_{\pm} = \sqrt{\lambda_{\pm}},$$

with mode vectors $r_{\pm} = (1, 2 - \lambda_{\pm})^T$. For the perturbation $\widehat{x}(0) = e_1, \widehat{x}'(0) = 0$,

$$\widehat{x}(t) = \sum_{\sigma=\pm} \frac{\cos(\omega_{\sigma}t)}{1 + (2 - \lambda_{\sigma})^2} r_{\sigma}.$$

Checks. At $t = 0$ the two modal projections must sum to $(1, 0)^T$.

Question difficulty allocation: $A = 8, B = 6, C = 2, D = 4$.

Question 4

Since $J = -\phi'$ and $J' = F$,

$$\phi''_- = f, \quad 0 < x < 1; \quad \phi''_+ = -2f, \quad 1 < x < 2.$$

Both ϕ and $J = -\phi'$ are continuous at $x = 1$. Solving with the endpoint conditions gives

$$\phi_-(x) = \frac{f}{2}x^2 + \left(\frac{V}{2} - \frac{f}{4}\right)x,$$

$$\phi_+(x) = -fx^2 + \left(\frac{V}{2} + \frac{11f}{4}\right)x - \frac{3f}{2}.$$

Now $J(0) = -V/2 + f/4$, so $J(0) = 0$ iff $V = f/2$. Then

$$\phi_- = \frac{f}{2}x^2, \quad \phi_+ = -fx^2 + 3fx - \frac{3f}{2}.$$

The maximum occurs where $\phi'_+ = -2fx + 3f = 0$, namely $x = 3/2$, and its value is $3f/4$. Also

$$J(0) = 0, \quad J(1) = -f, \quad J(2) = f.$$

These values determine the required sketches. **Question difficulty allocation:** $A = 16, B = 0, C = 0, D = 4$.