

Paper 3 - Solutions and Marking Guide

2023–2024 historical abilities

Question 1

In the order 1, 3, 2, 4 the weighted Laplacian is

$$L = \begin{pmatrix} c+1 & 0 & -c & -1 \\ 0 & c+1 & -1 & -c \\ -c & -1 & c+2 & -1 \\ -1 & -c & -1 & c+2 \end{pmatrix}.$$

The limiting networks give $C(0) = 1/3$ and $C(\infty) = 3$. KCL gives

$$v_2 = \frac{c+1}{c+3}, \quad v_4 = \frac{2}{c+3}.$$

The input current is

$$C(c) = c(1 - v_2) + (1 - v_4) = \frac{3c+1}{c+3}.$$

For a series connection with parameters c and $1/c$,

$$C_{\text{series}} = \frac{(c+3)(3c+1)}{2(5c^2+6c+5)}.$$

Question difficulty allocation: $A = 8$, $B = 4$, $C = 4$, $D = 4$.

Question 2

The vertex and edge counts are

$$V_n = 1 + 3 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}, \quad E_n = 3 + \cdots + 3^n = \frac{3^{n+1} - 3}{2}.$$

Each generation- j layer consists of 3^j equal conductors in parallel, so its resistance is $1/(3^j c_j)$. Hence

$$R_n = \sum_{j=1}^n \frac{1}{3^j c_j}, \quad C_n = R_n^{-1}.$$

For $c_j = 1$,

$$C_n = \frac{2}{1 - 3^{-n}}, \quad C_\infty = 2,$$

and the voltage in generation k is

$$v_k = \frac{3^{-k} - 3^{-n}}{1 - 3^{-n}}.$$

For $c_j = 1/j$,

$$R_\infty = \sum_{j \geq 1} \frac{j}{3^j} = \frac{3}{4}, \quad C_\infty = \frac{4}{3}.$$

For $c_j = j$,

$$R_\infty = \sum_{j \geq 1} \frac{1}{j 3^j} = \log \frac{3}{2}, \quad C_\infty = \frac{1}{\log(3/2)}.$$

Question difficulty allocation: $A = 4$, $B = 6$, $C = 4$, $D = 6$.

Question 3

For K_5 , $L = 5I - J$. The effective resistance between any two nodes is $2/5$, hence $C_{12} = 5/2$. A convenient orthonormal eigenbasis is

$$\begin{aligned} u_0 &= \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1)^T, \quad \lambda_0 = 0, \\ u_1 &= \frac{1}{\sqrt{2}}(1, -1, 0, 0, 0)^T, \quad u_2 = \frac{1}{\sqrt{6}}(1, 1, -2, 0, 0)^T, \\ u_3 &= \frac{1}{\sqrt{12}}(1, 1, 1, -3, 0)^T, \quad u_4 = \frac{1}{\sqrt{20}}(1, 1, 1, 1, -4)^T, \end{aligned}$$

all with eigenvalue 5 except u_0 . For $f = e_1 - e_2$, only the u_1 mode is present, so

$$x = \frac{1}{5}f = \left(\frac{1}{5}, -\frac{1}{5}, 0, 0, 0\right)^T.$$

The voltage difference is $2/5$, confirming $R = 2/5$ and $C = 5/2$. **Question difficulty allocation:** $A = 8, B = 4, C = 4, D = 4$.

Question 4

On $z = iy$, numerator and denominator in the logarithm have equal modulus, so $\phi = 0$. The finite singularities in $x > 0$ are a source $+m$ at $z = a$ and a sink $-m$ at $z = b$.

$$J_x - iJ_y = \frac{m}{2\pi} \left(\frac{1}{z-a} - \frac{1}{z+a} - \frac{1}{z-b} + \frac{1}{z+b} \right).$$

For $a = 1, b = 3$ and $z = iy$,

$$\boxed{J_x(0, y) = \frac{m}{\pi} \left(-\frac{1}{y^2 + 1} + \frac{3}{y^2 + 9} \right)}, \quad J_y = 0.$$

Its zeros are $y = \pm\sqrt{3}$, $J_x(0, 0) = -2m/(3\pi)$, and the tails are positive. The signed wall flux is zero because the source and sink strengths cancel:

$$\int_{-\infty}^{\infty} J_x(0, y) dy = 0.$$

Question difficulty allocation: $A = 12, B = 6, C = 0, D = 2$.