

Paper 4 - Solutions and Marking Guide

2020–2022 historical abilities

Question 1

The Laplacian is

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & -1 & -1 \\ -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

With $V_1 = 1, V_4 = 0$, symmetry and KCL give

$$V_2 = V_3 = \frac{3}{5}, \quad V_5 = V_6 = \frac{2}{5},$$

and $C_{\text{eff}} = 2(1 - 3/5) = 4/5$. For unit current, scale all voltages by $1/(4/5) = 5/4$. Thus the terminal voltage difference is $5/4$,

$$V_2 = V_3 = \frac{3}{4}, \quad V_5 = V_6 = \frac{1}{2},$$

and the current from 3 to 6 is $3/4 - 1/2 = 1/4$. **Question difficulty allocation:** $A = 8, B = 4, C = 4, D = 4$.

Question 2

The reduced Laplacian is

$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The three unit-voltage responses are

$$(1, 2/3, 1/3)^T, \quad (1/2, 1, 1/2)^T, \quad (1/3, 2/3, 1)^T.$$

The corresponding source divergences are $4/3, 1, 4/3$. Reciprocity follows from $f = Kx, \hat{f} = K\hat{x}$ and symmetry:

$$x^T \hat{f} = x^T K \hat{x} = (Kx)^T \hat{x} = \hat{x}^T f.$$

The Green matrix is

$$M = K^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Question difficulty allocation: $A = 8, B = 4, C = 4, D = 4$.

Question 3

Let L_3 be the triangle Laplacian. Since $A = L_3 + I$, symmetric vectors $(e, e)^T$ have eigenvalues of L_3 , while antisymmetric vectors $(e, -e)^T$ have eigenvalues of $L_3 + 2I$. The triangle eigenvalues are $0, 3, 3$, hence the prism eigenvalues are

$$0, 3, 3, 2, 5, 5.$$

Use any orthonormal triangle basis $e_0 = (1, 1, 1)/\sqrt{3}$ and two orthonormal vectors perpendicular to it, then normalise (e, e) and $(e, -e)$ by $1/\sqrt{2}$. For $x' = -Kx$, the constant mode is conserved. Every other component decays exponentially. The slowest non-constant rate is

$$2.$$

If the initial mean is zero, $x(t) \rightarrow 0$. **Question difficulty allocation:** $A = 8, B = 8, C = 0, D = 4$.

Question 4

For $z = iy$, the ratio $(a - z)/(a + z)$ has modulus 1, hence the wall is grounded. The circle equation is

$$(x - x_0)^2 + y^2 = r^2, \quad x_0 = \frac{a(1 + \rho^2)}{1 - \rho^2}, \quad r = \frac{2a\rho}{1 - \rho^2}.$$

On it, $\Re \log((a - z)/(a + z)) = \log \rho$, so $\phi = V$.

$$J_x - iJ_y = -h'(z) = \frac{2aV}{\log \rho (a^2 - z^2)}.$$

On the wall,

$$J_x(0, y) = \frac{2aV}{\log \rho (a^2 + y^2)}, \quad J_y = 0.$$

Its magnitude is largest at $y = 0$. The total outward current is

$$-\int_{-\infty}^{\infty} J_x dy = \boxed{-\frac{2\pi V}{\log \rho}} > 0.$$

If $V/\log \rho \rightarrow -m/(2\pi)$ and $\rho \rightarrow 0$, the circle shrinks to $z = a$ and the logarithmic singularity has source strength m . **Question**

difficulty allocation: $A = 8$, $B = 4$, $C = 4$, $D = 4$.