

Paper 5 - Solutions and Marking Guide

Seven-year gap-closing paper

Question 1

There are 8 edges and 5 vertices. An edge-by-vertex incidence matrix is 8×5 , rank 4, right-null dimension 1 and left-null dimension 4. The Laplacian in node order 0, 1, 2, 3, 4 is

$$L = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{pmatrix}.$$

An orthogonal eigenbasis before normalisation is

$$(1, 1, 1, 1, 1), \quad (0, 1, 0, -1, 0), \quad (0, 0, 1, 0, -1), \\ (0, 1, -1, 1, -1), \quad (-4, 1, 1, 1, 1),$$

with eigenvalues 0, 3, 3, 5, 5. Normalise by $\sqrt{5}, \sqrt{2}, \sqrt{2}, 2, \sqrt{20}$ respectively. **Question difficulty allocation:** $A = 8, B = 4, C = 4, D = 4$.

Question 2

There are 10 edges and 7 vertices, so the incidence matrix is 10×7 , rank 6, right-null dimension 1 and left-null dimension 4. For hitting U or V before T , harmonic equations give

$$P_A = \frac{32}{81}, \quad P_B = \frac{58}{81}, \quad P_C = \frac{38}{81}, \quad P_X = \frac{19}{81}.$$

Thus B is best. Starting at T and conditioning on the first step,

$$P_T = \frac{P_A + P_C + P_X}{3} = \frac{89}{243}.$$

After closing V , solve with boundary values $P_A = 0, P_C = 1$. The internal values are $P_B = P_U = 1/2, P_T = 3/5, P_X = 4/5$. After leaving A ,

$$P(\text{hit } C \text{ before return } A) = \frac{1/2 + 1 + 3/5}{3} = \frac{7}{10}.$$

Question difficulty allocation: $A = 4, B = 6, C = 6, D = 4$.

Question 3

The cycle Laplacian is

$$K = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix},$$

and

$$\widehat{K} = \begin{pmatrix} 2 & -2 & 0 & -2 \\ -2 & 8 & -2 & 0 \\ 0 & -2 & 2 & -2 \\ -2 & 0 & -2 & 8 \end{pmatrix}.$$

The half-turn is an involution and preserves all masses and springs, so it commutes with $\widehat{K}$. The eigenpairs are

$$0 : (2, 1, 2, 1), \quad 2 : (-1, 0, 1, 0), \quad 8 : (0, -1, 0, 1), \quad 10 : (-1, 2, -1, 2).$$

Therefore the natural frequencies are

$$0, \sqrt{2}, 2\sqrt{2}, \sqrt{10}.$$

The zero-frequency vector is the rigid translation in mass-scaled coordinates. **Question difficulty allocation:** $A = 8, B = 6, C = 2, D = 4$.

Question 4

For $|x| < 1$ approached from above,

$$\sqrt{x^2 - 1} = i\sqrt{1 - x^2},$$

and $|x + i\sqrt{1 - x^2}| = 1$, so $\phi = 0$. At infinity

$$h(z) \sim -\frac{m}{2\pi} \log(2z),$$

which supplies the net far-field current associated with strength m . Since

$$\frac{d}{dz} \log(z + \sqrt{z^2 - 1}) = \frac{1}{\sqrt{z^2 - 1}},$$

$$\boxed{J_x - iJ_y = \frac{m}{2\pi\sqrt{z^2 - 1}}}.$$

On the electrode,

$$J_x = 0, \quad J_y = \frac{m}{2\pi\sqrt{1 - x^2}},$$

so the inward flux into the conductor is

$$\int_{-1}^1 J_y dx = \frac{m}{2}.$$

For $x > 1$, $J_y = 0$ and

$$J_x = \frac{m}{2\pi\sqrt{x^2 - 1}} \sim \frac{m}{2\pi\sqrt{2(x - 1)}}.$$

Question difficulty allocation: $A = 12$, $B = 4$, $C = 0$, $D = 4$.