

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2020

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

An Introduction to Applied Maths

Date: 18th May 2020

Time: 09.00am – 11.00am (BST)

Time Allowed: 2 Hours

Upload Time Allowed: 30 Minutes

This paper has 4 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

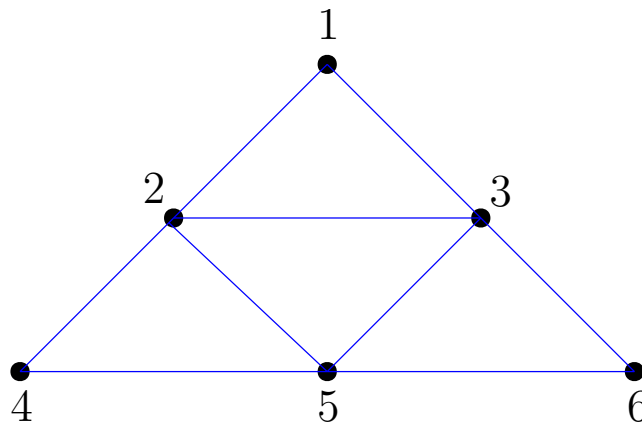
Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

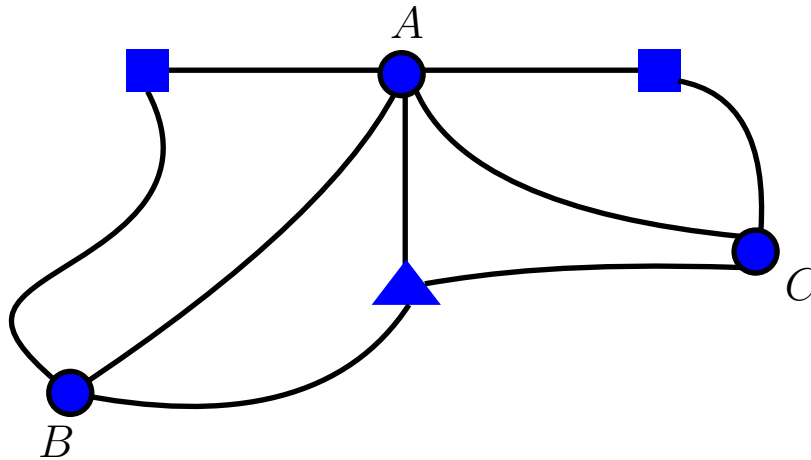
1. Consider the electrical circuit shown in the Figure comprising 6 nodes connected by 9 resistors each of which has unit conductance.



- (a) Using the ordering of nodes indicated in the Figure, find the Laplacian matrix of the graph associated with this circuit. (4 marks)
- (b) Node 1 is set at unit voltage, node 4 is grounded, and Kirchhoff's current law holds at all other nodes. Find the voltages at nodes 2, 3, 5 and 6. (8 marks)
- (c) Suppose a unit current enters the circuit at node 1 and exits the circuit at node 4. Find the current into node 6 from node 3. (8 marks)

(Total: 20 marks)

2. This question concerns a simple random journey around a train network with 6 stations connected with train lines as shown in this Figure:

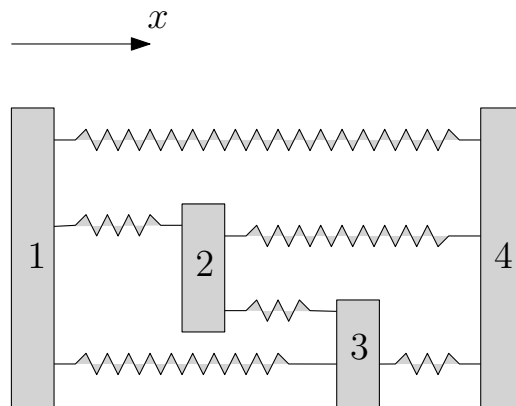


At each station the random traveller picks at random a train line to take out of that station with all lines having equal probability, including possibly going back on the train line just taken.

- (a) Suppose the random traveller can start at any of the 3 stations A , B or C . Also, he wants to maximize his probability of reaching one of the two square stations before he reaches the triangular station. Determine which of the 3 stations A , B or C he should start at, and find the associated probability of reaching a square station before the triangular station. (6 marks)
- (b) Suppose the random traveller starts at the triangular station. What is the probability that he reaches either of the two square stations before returning to the triangular station? (6 marks)
- (c) The square station on the right in the Figure is closed for engineering works so that the two train lines connecting it to stations A and C are out of service. Find the probability of a random traveller starting at station A reaching station B before returning to station A . (8 marks)

(Total: 20 marks)

3. Consider the system of 4 masses linked by springs as shown in the Figure:



All springs have unit spring constant (even though, in the Figure, they have different lengths). All masses have unit mass (even though, in the Figure, they have different sizes). The masses can only experience displacements to the left and right in the x direction shown.

- (a) By treating this spring-mass system as a graph with 4 nodes (the masses) and 6 edges (the springs) find the 4-by-4 generalized Laplacian matrix associated with the internal spring forces in this system. (4 marks)
- (b) Suppose mass 1 is held fixed and let $\mathbf{X}(t) = (X_2(t), X_3(t), X_4(t))^T$ be the 3-vector of displacements under free oscillation (no external forces) of masses 2, 3 and 4. On letting

$$\mathbf{X}(t) = e^{i\omega_j t} \mathbf{e}_j, \quad j = 1, 2, 3,$$

where $\mathbf{e}_j$ is a constant vector, find the three possible natural frequencies ω_j and the associated vectors $\mathbf{e}_j$ of free oscillation of this system. (6 marks)

- (c) Consider now an equilibrium configuration in which mass 1 is held fixed in place so it cannot be displaced, while an external force $+1$ is imposed on mass 4. There are no external forces on masses 2 and 3. Let the associated equilibrium displacements of masses 2, 3, and 4 be denoted by the 3-vector Φ and write

$$\Phi = \sum_{j=1}^3 a_j \mathbf{e}_j,$$

where the vectors $\{\mathbf{e}_j | j = 1, 2, 3\}$ are those found in part (b). Find the coefficients a_1, a_2 and a_3 .

(6 marks)

- (d) Hence find the displacements of masses 2, 3 and 4 for the equilibrium in part (c). (4 marks)

(Total: 20 marks)

4. Let D be the first quadrant $x > 0, y > 0$ in a two-dimensional (x, y) plane and suppose that it is an electrical conductor with unit conductivity. Suppose the voltage distribution $\phi(x, y)$ associated with a current in D is given by the function

$$\phi(x, y) = \operatorname{Re}[h(z)],$$

where $z = x + iy$ and

$$h(z) = -\frac{m}{2\pi} \log \left[\frac{z^2 - i}{z^2 + i} \right].$$

The notation $\operatorname{Re}[\cdot]$ means take the real part of the complex-valued quantity in square brackets.

- (a) Verify that $\nabla^2 \phi(x, y) = 0$ everywhere in D except at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. (5 marks)
- (b) Show that the boundary $y = 0$ is grounded, i.e. show that $\phi(x, 0) = 0$. (2 marks)
- (c) Show that the boundary $x = 0$ is grounded, i.e. show that $\phi(0, y) = 0$. (2 marks)
- (d) If the associated current density vector is written as $\mathbf{j} = (j_x, j_y)$ find the y -component of the current density distribution on the boundary $y = 0, x \geq 0$ as a function of x (that is, find the quantity $j_y(x, 0)$ as a function of x). (5 marks)
- (e) Hence show that the total current passing across the boundary $y = 0, x \geq 0$ out of the conductor D is equal to $m/2$. (6 marks)

(Total: 20 marks)

M40007 Summer 2020 sample solutions

General note: The following solutions are indicative only; they are not the only way to solve the problems, merely suggested ways.

1 (a) Using the usual rules for the construction of a discrete Laplacian, and using the numbering of nodes given in the figure, we find

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 & 0 \\ -1 & -1 & 4 & 0 & -1 & -1 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}.$$

4 marks (seen similar)

(b) The equations of current balance are

$$\mathbf{K}\mathbf{x} = \mathbf{f} = \begin{bmatrix} f_0 \\ 0 \\ 0 \\ -f_0 \\ 0 \\ 0 \end{bmatrix}$$

where $\mathbf{f}$ is the vector containing the divergence of the currents at the nodes. The equations from this system coming from the nodes at which Kirchhoff's current law holds yield

$$2\phi_6 = \phi_3 + \phi_5 \quad (1)$$

$$4\phi_3 = \phi_1 + \phi_2 + \phi_5 + \phi_6 = 1 + \phi_2 + \phi_5 + \phi_6 \quad (2)$$

$$4\phi_2 = \phi_1 + \phi_3 + \phi_5 + \phi_4 = \phi_1 + \phi_3 + \phi_5 \quad (3)$$

$$4\phi_5 = \phi_4 + \phi_6 + \phi_2 + \phi_3 = \phi_6 + \phi_2 + \phi_3. \quad (4)$$

These can be solved by hand. Equations (1) and (3) imply

$$\phi_6 = 2\phi_2 - \frac{1}{2}. \quad (5)$$

Equations (2) and (4) imply

$$\phi_5 = \phi_3 - \frac{1}{5}. \quad (6)$$

Substitution for ϕ_6 from (4) and for ϕ_5 from (6) in (1) leads to

$$2 \left[2\phi_2 - \frac{1}{2} \right] = \phi_3 + \phi_3 - \frac{1}{5} \quad (7)$$

or

$$\phi_3 - 2\phi_2 = -\frac{2}{5}. \quad (8)$$

This is one equation between ϕ_2 and ϕ_3 . Another follows on substituting for ϕ_6 from (4) and for ϕ_5 from (6) in (4):

$$4 \left[\phi_3 - \frac{1}{5} \right] = \left[2\phi_2 - \frac{1}{2} \right] + \phi_2 + \phi_3 \quad (9)$$

or

$$\phi_3 - \phi_2 = \frac{1}{10}. \quad (10)$$

Solving (8) and (10) gives

$$\phi_2 = \frac{1}{2}, \quad \phi_3 = \frac{3}{5}. \quad (11)$$

Back substitution then gives

$$\phi_5 = \frac{2}{5}, \quad \phi_6 = \frac{1}{2}. \quad (12)$$

8 marks (seen similar)

(c) The first step is to find the net current into the circuit at node 1. Given that all conductances are unity, this is

$$(1 - \phi_3) + (1 - \phi_2) = \frac{2}{5} + \frac{1}{2} = \frac{9}{10}. \quad (13)$$

To make this input current unity, as required in the question, it is clear that we need to multiply all voltages just found by $10/9$; the linearity of the system allows us to do this. Then the current into node 6 from node 3 is

$$\frac{10}{9} (\phi_3 - \phi_6) = \frac{10}{9} \left(\frac{3}{5} - \frac{1}{2} \right) = \frac{1}{9}.$$

8 marks (seen similar)

2 (a) A useful observation is that, by the symmetry of the network, the probability of reaching a square station before the triangular station from either of the stations B and C must be equal. Call this probability ψ . Let ϕ be the probability of reaching a square station before the triangular station when starting from node A . The same probability starting at a square station is 1, and is 0 if the walker starts at the triangular station.

By the basic laws of probability, we know these probabilities at the nodes are harmonic potentials given by the average of the potentials at their connected neighbours. Therefore the equations for the two unknown probabilities are

$$\begin{aligned} 5\phi &= 2 + 2\psi, \\ 3\psi &= 1 + \psi. \end{aligned} \tag{14}$$

These are easily solved to find

$$\phi = \frac{8}{13}, \quad \psi = \frac{7}{13}.$$

Since $\phi > \psi$ the walker should therefore start his walk at station A to maximize the probability of reaching a square station before the triangular station.

6 marks (seen similar)

(b) This question is asking for the escape probability to the square stations from the triangular station which requires the solution to a potential problem for the probabilities when the probability at the triangular node is now unity (as opposed to 0 as in part (a)), and the probabilities starting at either of the square nodes is 0 (as opposed to 1 as in part (a)).

The key observation is that the solution to this problem can be obtained by

- subtracting unity from all the potentials in the solution from part (a);
- multiplying all potentials by -1 .

After this transformation, the required escape probability is

$$\frac{C_{\text{eff}}}{3} \tag{15}$$

where, in the electric circuit analogy, C_{eff} is the net current out of the “triangular station” and the denominator is the sum of the conductances of edges out of it (i.e the number of edges out of this station given that the conductances are all

unity). The new “voltage” at the triangular station is 1 and that at stations B and C is $6/13$ while that at station A is $5/13$. Then

$$C_{\text{eff}} = 2 \left(1 - \frac{6}{13} \right) + \left(1 - \frac{8}{13} \right) = \frac{22}{13}. \quad (16)$$

Hence the required escape probability is

$$\frac{22}{39}. \quad (17)$$

6 marks (seen similar)

(c) The modified network, with two edges down, is equivalent to the circuit shown in the figure below. We then require the escape probability from station A to B which can be computed using

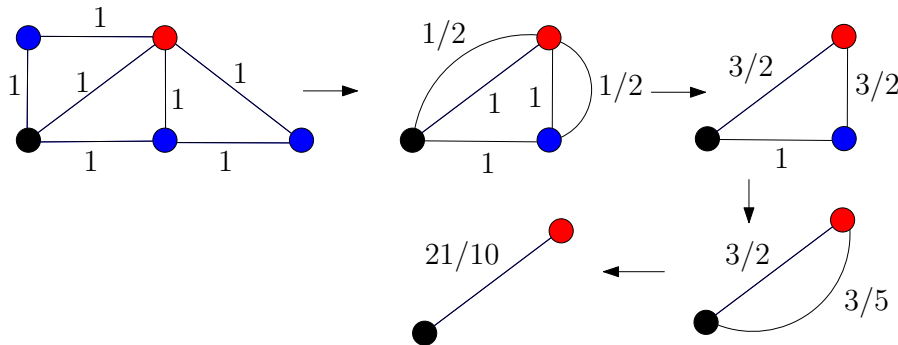
$$\frac{C_{\text{eff}}}{4} \quad (18)$$

where the denominator is the number of (unit conductance) edges out of station A and C_{eff} is, in the circuit analogy, the effective conductance of the circuit when station A is at unit voltage and station B is grounded. This quantity is most easily calculated by reducing to a series of equivalent circuits as shown in the figure. In producing this sequence of diagrams we have repeatedly used the rules for two conductors in series:

$$C_{\text{eff}} = \frac{c_1 c_2}{c_1 + c_2} \quad (19)$$

and two conductors in parallel:

$$C_{\text{eff}} = c_1 + c_2 : \quad (20)$$



(edge conductances are shown). In this way, we find

$$\mathcal{C}_{\text{eff}} = \frac{21}{10}. \quad (21)$$

The required escape probability is therefore

$$\frac{\mathcal{C}_{\text{eff}}}{4} = \frac{21}{40}. \quad (22)$$

8 marks (seen similar)

3 (a) Using the usual rules for the construction of a discrete Laplacian matrix, we find

$$\mathbf{K} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

4 marks (seen similar)

(b) If node 1 is grounded the reduced Laplacian becomes

$$\hat{\mathbf{K}} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

A key observation is that this real symmetric matrix is a 3-by-3 circulant matrix so, using results on such matrices derived in the course, the eigenvectors can immediately be deduced to be

$$\begin{bmatrix} 1 \\ \omega^m \\ \omega^{2m} \end{bmatrix}, \quad m = 0, 1, 2,$$

where

$$\omega = e^{2\pi i/3}.$$

The eigenvalues λ_1, λ_2 and λ_3 can now be found by direct substitution into $\mathbf{K}\mathbf{x} = \lambda\mathbf{x}$:

$$\begin{aligned} \lambda_1 &= 3 - \omega^0 - \omega^0 = 1, \\ \lambda_2 &= 3 - \omega^1 - \omega^2 = 3 - 2 \cos(2\pi/3) = 4, \\ \lambda_3 &= 3 - \omega^2 - \omega^4 = 3 - 2 \cos(2\pi/3) = 4. \end{aligned}$$

On normalizing these eigenvectors so that $\bar{\mathbf{e}}_m^T \mathbf{e}_n = \delta_{mn}$ they become

$$\mathbf{e}_m = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \omega^m \\ \omega^{2m} \end{bmatrix}, \quad m = 0, 1, 2.$$

Additional notes:

- A direct calculation of eigenvectors and eigenvalues, via determinant methods, is also acceptable, although the route via circulant matrices is faster.
- Some students may prefer working with real eigenvectors. This is fine. They would be $\mathbf{e}_1$ and the real and imaginary parts of $\mathbf{e}_2$, i.e.,

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ \sqrt{3}/2 \\ -\sqrt{3}/2 \end{bmatrix}.$$

These are orthogonal (and can be made orthonormal). The coefficients in part (c) will be different, of course. The solution in part (d) should be the same.

- Also, since the eigenvalue 4 is of multiplicity 2, and since the question did not ask the students to use an orthonormal set of eigenvectors, it is also possible that the students will find other sets of eigenvectors given by, for example,

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \hat{\mathbf{e}}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

in which case the relevant coefficients are

$$a_1 = \frac{1}{\sqrt{3}}, \quad a_2 = -\frac{\sqrt{2}}{3} = a_3.$$

6 marks (seen similar)

(c) The equation for the equilibrium displacements of masses 1,2 and 3 i.e.,

$$\Phi = (\phi_1, \phi_2, \phi_3)^T$$

under the imposed external force is

$$\hat{\mathbf{K}}\Phi = \mathbf{f} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If we write

$$\Phi = \sum_{m=1}^3 a_m \mathbf{e}_m,$$

where $\{\mathbf{e}_m\}$ are defined above then on substitution we find

$$\hat{\mathbf{K}}\Phi = \hat{\mathbf{K}} \sum_{m=1}^3 a_m \mathbf{e}_m = \sum_{m=1}^3 \lambda_m a_m \mathbf{e}_m = \mathbf{f}.$$

On multiplying the previous equation by $\bar{\mathbf{e}}_j^T$ we find

$$\sum_{m=1}^3 \lambda_m a_m \bar{\mathbf{e}}_j^T \mathbf{e}_m = \bar{\mathbf{e}}_j^T \mathbf{f}.$$

Using the orthonormality of the eigenvectors $\bar{\mathbf{e}}_j^T \mathbf{e}_n = \delta_{jn}$, this results in

$$a_1 = \frac{1}{\sqrt{3}}, \quad a_2 = \frac{\omega}{4\sqrt{3}}, \quad a_3 = \frac{\omega^2}{4\sqrt{3}}.$$

6 marks (seen similar)

(d) The final solution is then

$$\Phi = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\omega}{12} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} + \frac{\omega^2}{12} \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/2 \end{bmatrix}.$$

It is easy to check this final result by direct solution of the linear system.

4 marks (seen similar)

4(a): Since

$$\phi = \frac{1}{2} \left[h(z) + \overline{h(z)} \right] \quad (23)$$

to compute $\nabla^2 \phi$ we can compute

$$\nabla^2 h(z)$$

and add its complex conjugate. We note that $h(z)$ is differentiable everywhere except at

$$z^2 = \pm i.$$

These are the points

$$z = e^{in\pi/4}, \quad n = 1, 3, 5, 7.$$

and only the case $n = 1$ is in D , i.e., the point

$$e^{i\pi/4} = \frac{1}{\sqrt{2}}(1 + i) \quad (24)$$

which is the point $(1/\sqrt{2}, 1/\sqrt{2})$. Now, outside this point, we have

$$\frac{\partial h(z)}{\partial x} = h'(z) \frac{\partial z}{\partial x} = h'(z)$$

and

$$\frac{\partial^2 h(z)}{\partial x^2} = h''(z) \frac{\partial z}{\partial x} = h''(z). \quad (25)$$

Similarly,

$$\frac{\partial h(z)}{\partial y} = h'(z) \frac{\partial z}{\partial y} = ih'(z)$$

and

$$\frac{\partial^2 h(z)}{\partial y^2} = ih''(z) \frac{\partial z}{\partial y} = -h''(z). \quad (26)$$

It is clear on addition of (25) and (26) that

$$\nabla^2 h(z) = 0 \quad (27)$$

and hence that

$$\nabla^2 h(z) + \nabla^2 \overline{h(z)} = \nabla^2 \left[h(z) + \overline{h(z)} \right] = 2\nabla^2 \phi = 0. \quad (28)$$

Thus ϕ is harmonic at all points in D exterior to the point $(1/\sqrt{2}, 1/\sqrt{2})$.

5 marks (seen similar)

(b) On $y = 0$ we have $z^2 = x^2$ and

$$h(z) = -\frac{m}{2\pi} \log \left[\frac{x^2 - i}{x^2 + i} \right].$$

Hence

$$\phi = \frac{1}{2} \left[h(z) + \overline{h(z)} \right] = -\frac{m}{4\pi} \log \left[\frac{x^2 - i}{x^2 + i} \right] - \frac{m}{4\pi} \log \left[\frac{x^2 + i}{x^2 - i} \right] = -\frac{m}{4\pi} \log 1 = 0. \quad (29)$$

2 marks (seen similar)

(c) On $x = 0$ we have $z^2 = -y^2$ and

$$h(z) = -\frac{m}{2\pi} \log \left[\frac{-y^2 - i}{-y^2 + i} \right].$$

Hence

$$\phi = \frac{1}{2} \left[h(z) + \overline{h(z)} \right] = -\frac{m}{4\pi} \log \left[\frac{-y^2 - i}{-y^2 + i} \right] - \frac{m}{4\pi} \log \left[\frac{-y^2 + i}{-y^2 - i} \right] = -\frac{m}{4\pi} \log 1 = 0. \quad (30)$$

2 marks (seen similar)

(d) We know that

$$j_x - ij_y = -\hat{c}h'(z),$$

where $\hat{c} = 1$ is the conductivity. Hence

$$j_x - ij_y = \frac{m}{2\pi} \left[\frac{2z}{z^2 - i} - \frac{2z}{z^2 + i} \right] = \frac{2imz}{\pi(z^4 + 1)}. \quad (31)$$

On $y = 0$ we have $z = x$

$$j_x - ij_y = \frac{2imx}{\pi(x^4 + 1)}. \quad (32)$$

implying that, on this line,

$$j_y = -\frac{2mx}{\pi(x^4 + 1)}. \quad (33)$$

5 marks (seen similar)

(e) The total current, in the positive y direction, through the line $x > 0, y = 0$ is given by the integral

$$\int_0^\infty j_y dx = - \int_0^\infty \frac{2mx}{\pi(x^4 + 1)} dx.$$

Introduce the change of variable $\chi = x^2$, then

$$\int_0^\infty j_y dx = - \int_0^\infty \frac{m}{\pi(\chi^2 + 1)} d\chi.$$

Then introduce the substitution $\chi = \tan \theta$ so that

$$\int_0^\infty j_y dx = - \int_0^{\pi/2} \frac{m}{\pi(\tan^2(\theta) + 1)} \sec^2 \theta d\theta = - \int_0^{\pi/2} \frac{m}{\pi} d\theta = -\frac{m}{2}.$$

Thus the total current *leaving* D through this part of the boundary is $m/2$.

6 marks (seen similar)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	Question	Comments for Students	
MATH40007	1	A very good performance overall. Many students missed the idea that the answer to part C can be obtained by a simple rescaling of the answer to part B, which was a shame.	
MATH40007	2	This was mostly done very well, a good number of students got all marks. I marked all parts generously. As long as I could see evidence for using the right method, I would award some marks. Part a) was done either by elementary considerations or by using potential theory. Part b), which can be done in a single line, was mostly done using potential theory. Common mistakes were to use the wrong relation between p_{esc} and C_{eff} . I would think this part was the most difficult one. Part c) was generally done better, with most students using potential theory.	
MATH40007	3	Part (a) was successfully answered by almost all students. Almost no student noticed that the laplacian was circulant (only 2-3 students noticed) and so nobody used this property to get easily eigenvalues and eigenvectors. The vast majority of students computed eigenvalues and eigenvectors correctly. Depending on your choice of eigenvectors, you did not necessarily end up with an orthonormal basis which means that you could not use this in question (c) and (d). Some students did not check orthogonality and failed because of this. A lot of students got confused when asked to produce 3 natural frequencies as 4 was a degenerate eigenvalue. Taking the square roots would lead to ± 1 and ± 2 (to be used twice). Students who chose arbitrarily 3 numbers out of these 4 were penalized for inconsistency, why drop -1 and keep -2 for instance? Students who wrote that the natural frequencies were 1,2,2 or $\pm 1, \pm 2, \pm 2$ both got full marks. The level of overall justification and clarity of the answers was generally very poor. A large proportion of students jotted down the final answers without showing their work. Overall, the students should justify and explain their reasoning a lot better. Some students did this very well.	

MATH40007	Most students worked out the laplacian of the voltage distribution correctly; only few students discussed correctly the potentially problematic points (where the log was not defined). Checking that the provided points led to a problem was not enough and all problematic points needed to be discussed. Part (b) and (c) showed that some students need to revise fundamental notion like manipulating complex numbers and the definition of the complex logarithm. Too many students provided long and convoluted proofs to these simple questions which relied mainly on the modulus of a ratio of complex numbers. Many students made avoidable algebra mistakes which led in most cases to the wrong sign. In part (e), many students failed because of the sign mistake made in (d). Students were in general penalized for not explaining clearly their reasoning (and the signs of the current they were obtaining). Unclear answers were penalized. As a general comment, a lot of students are jumping straight and writing some maths which may be clear to them, i.e. a lot of students did not explain enough their reasoning. Your proofs should be limpid to the marker, as nothing should be left to interpretation. In general, leaving things unclear will not play in your favor. Even if it seems sort of trivial, students must always show their work (for instance solving the integral in (e) in one step without explaining the substitutions employed is not satisfactory).	
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