

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2025

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

An Introduction to Applied Mathematics

Date: Wednesday, May 14, 2025

Time: Start time 14:00 – End time 16:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Consider the matrix

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$

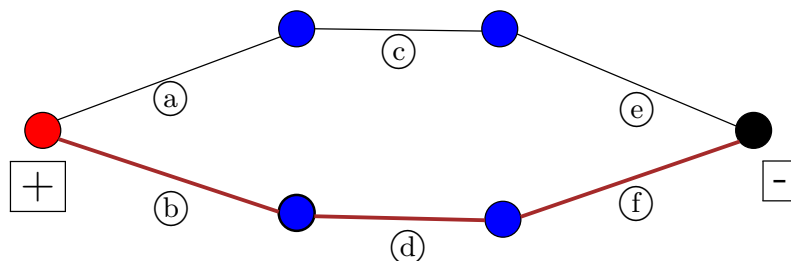
- (i) Find the rank of the matrix. (1 mark)
(ii) Find its eigenvalues. (3 marks)

- (b) Consider the matrix

$$\begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

- (i) Find the rank of this matrix. (1 mark)
(ii) Find its eigenvalues. (4 marks)

- (c) Consider the electric circuit shown in this figure:



Edges (a), (c) and (e) have conductance $C > 0$, and edges (b), (d) and (f) have conductance $D > 0$. Let $\mathbf{w} = (w_a, w_b, w_c, w_d, w_e, w_f)^T$ be a vector of currents in the edges (a)–(f). An external current $+1$ is fed into the circuit at node $\boxed{+}$, and this current leaves the circuit at node $\boxed{-}$, all other nodes have no external current source. Thomson's principle says that, among all currents satisfying these current divergence conditions at the nodes, the current $\mathbf{w}_*$ actually observed in the circuit is the one that minimizes the quantity

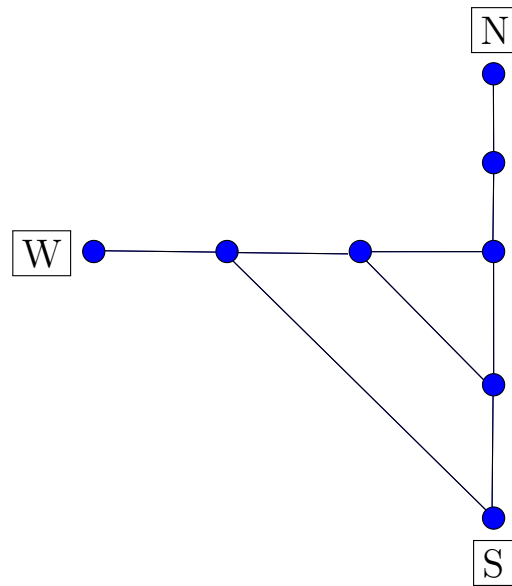
$$\tilde{\mathcal{E}}(\mathbf{w}) = \mathbf{w}^T \mathbf{C}^{-1} \mathbf{w},$$

where $\mathbf{C}$ is the diagonal 6-by-6 matrix of edge conductances.

- (i) Use Thomson's principle to find $\mathbf{w}_*$. (4 marks)
(ii) Use the method of equivalent circuits to find the effective conductance of the circuit. (3 marks)
(iii) Use your result in part (ii) to check your answer to part (i). (4 marks)

(Total: 20 marks)

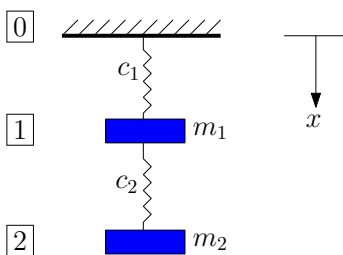
2. Consider the graph shown in the figure:



- (a) Let $\mathbf{A}$ be any incidence matrix of this graph.
- Find the dimensions of $\mathbf{A}$. (There is no need to write down $\mathbf{A}$). (1 mark)
 - Find the dimensions of the right and left null spaces of $\mathbf{A}$. (2 marks)
- (b) A random walker takes a journey around this graph choosing to traverse each edge at any given node with equal probability. Supposing that a journey starts at node $\boxed{W}$, find the probability that the random walker reaches node $\boxed{S}$ before returning to node $\boxed{W}$. (8 marks)
- (c) The random walker now starts another journey at node $\boxed{N}$. Find the probability that the random walker hits node $\boxed{W}$ before node $\boxed{S}$. (9 marks)

(Total: 20 marks)

3. Consider the spring-mass system shown on the figure where the masses can only displace in the x direction whose positive direction is indicated:



The ceiling is node $\boxed{0}$ and it is fixed (it cannot be displaced). Two masses, $m_1 = m_2 = 1$ are nodes $\boxed{1}$ and $\boxed{2}$; they are connected by springs with spring constants: $c_1 = c_2 = 1$. No gravity is active. Mass $\boxed{1}$ is free of external force, $f_1 = 0$, but an external force $f_2 = 1$ acts on mass $\boxed{2}$.

- (a) Using the natural ordering, $\boxed{0}$, $\boxed{1}$, $\boxed{2}$, let

$$\mathbf{x}_e = \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}$$

be the vector of x -displacements at the nodes in equilibrium under the effect of the external forces. Find $\mathbf{x}_e$. (4 marks)

- (b) Find the reaction force at the ceiling when the system is in equilibrium. (2 marks)
- (c) Let t denote time. Without changing the external forces on the masses $\boxed{1}$ or $\boxed{2}$, suppose that the equilibrium in part (a) is perturbed so that the vector of displacements $\mathbf{x}(t)$ is now

$$\mathbf{x}(t) = \mathbf{x}_e + \begin{pmatrix} 0 \\ \hat{\mathbf{x}}(t) \end{pmatrix}.$$

Find the two oscillation frequencies $\omega_{\pm}$ of the perturbation $\hat{\mathbf{x}}(t)$. (6 marks)

- (d) Given the initial conditions

$$\hat{\mathbf{x}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{d\hat{\mathbf{x}}}{dt}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for the perturbation at time $t = 0$, show that

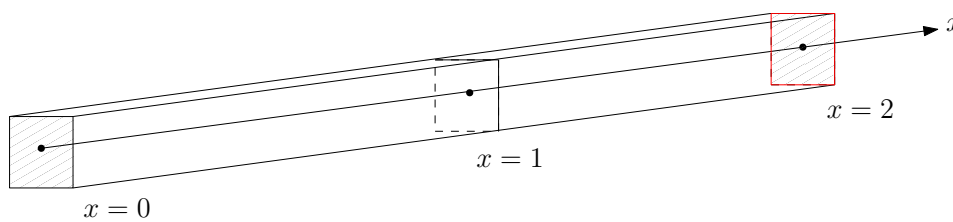
$$\hat{\mathbf{x}}(t) = \frac{(2 - \omega_-^2) \cos(\omega_+ t)}{\omega_+^2 - \omega_-^2} \begin{pmatrix} 1 \\ 2 - \omega_+^2 \end{pmatrix} - \frac{(2 - \omega_+^2) \cos(\omega_- t)}{\omega_+^2 - \omega_-^2} \begin{pmatrix} 1 \\ 2 - \omega_-^2 \end{pmatrix}. \quad (8 \text{ marks})$$

(Total: 20 marks)

4. Consider current flow in a continuous 1D conductor, of uniform conductivity $C(x) = 1$, occupying the interval $0 \leq x \leq 2$ as indicated in the figure. The current is driven in the conductor by a current source density distribution given by

$$F(x) = \begin{cases} -f_0, & 0 < x < 1, \\ +f_0, & 1 < x < 2, \end{cases}$$

where f_0 is a positive constant. The end of the conductor at $x = 0$ is grounded, the other end is held at voltage V , where V is a positive constant, so that $\phi(0) = 0, \phi(2) = V$ where $\phi(x)$ is the voltage distribution in the conductor.



- (a) Since the current source density distribution is discontinuous at $x = 1$ it is natural to decompose the voltage distribution $\phi(x)$ in the conductor as

$$\phi(x) = \begin{cases} \phi_-(x), & 0 \leq x \leq 1, \\ \phi_+(x), & 1 \leq x \leq 2. \end{cases}$$

Write down the differential equations and boundary conditions satisfied by $\phi_-(x)$ and $\phi_+(x)$. (3 marks)

- (b) Find the two functions $\phi_-(x)$ and $\phi_+(x)$. (8 marks)
- (c) It is now additionally stipulated that the current at $x = 0$ must be zero. Using your solution to part (b), find the relationship between V and f_0 that must hold in order for this to be the case. (3 marks)
- (d) Write down the solution for $\phi(x)$ corresponding to the situation in part (c). (2 marks)
- (e) Sketch a graph of the voltage distribution $\phi(x)$ for $0 \leq x \leq 2$ found in part (d). (4 marks)

(Total: 20 marks)

- The set of linearly independent vectors

$$\mathbf{x}_k = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \vdots \\ \omega^{(n-1)k} \end{pmatrix}, \quad \omega = e^{2\pi i/n}, \quad k = 0, 1, \dots, n-1,$$

are complex-valued eigenvectors of an n -by- n circulant matrix.

- The matrix

$$\mathbf{K}_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}, \quad n \geq 2$$

has orthonormal eigenvectors $\{\Phi_m\}$ and corresponding eigenvalues $\{\lambda_m\}$ given by

$$\Phi_m = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{m\pi}{n+1}\right) \\ \sin\left(\frac{2m\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nm\pi}{n+1}\right) \end{pmatrix}, \quad \lambda_m = 2 - 2\cos\left(\frac{\pi m}{n+1}\right), \quad m = 1, \dots, n$$

- The current $W(x)$ in a 1D conductor with conductivity distribution $C(x)$ along the x -axis is related to the voltage distribution $\Phi(x)$ and the external current source density distribution $F(x)$ by

$$W(x) = -C(x) \frac{d\Phi(x)}{dx}, \quad \frac{dW}{dx} = F(x).$$

- The current density vector $\mathbf{j} = (J^{(x)}(x, y), J^{(y)}(x, y))^T$ in a 2D conductor with uniform conductivity $\hat{c}$ located in an (x, y) -plane is related to the voltage distribution $\Phi(x, y)$ and the external current source density distribution $F(x, y)$ in the conductor by

$$\mathbf{j}(x, y) = \begin{pmatrix} J^{(x)}(x, y) \\ J^{(y)}(x, y) \end{pmatrix} = -\hat{c} \nabla \Phi(x, y), \quad \nabla \cdot \mathbf{j} = F(x, y).$$

The voltage potential $\Phi(x, y)$ in a two-dimensional conductor of uniform conductivity $\hat{c}$ can be written in terms of an analytic function $h(z)$,

$$\Phi(x, y) = \operatorname{Re}[h(z)], \quad z = x + iy,$$

where $\operatorname{Re}[\cdot]$ denotes the real part of the quantity in square brackets. Also,

$$J^{(x)}(x, y) - iJ^{(y)}(x, y) = -\hat{c} \frac{dh}{dz}.$$

If, near some point z_0 inside the conductor, $h(z)$ has a local form

$$h(z) = -\frac{m}{2\pi} \log(z - z_0) + \hat{h}(z),$$

where $\hat{h}(z)$ is non-singular at z_0 , then we say there is a point source of current of strength m at z_0 .

1. (a) This matrix is recognized as

$$\mathbf{K}_3 + \mathbf{I}_3,$$

where $\mathbf{K}_3$ is given on the exam handout (with its eigenvectors and eigenvalues) and $\mathbf{I}_3$ is the 3-by-3 identity matrix. It is clear that this matrix shares the same eigenvectors as $\mathbf{K}_3$ and therefore has eigenvalues

$$2 - 2 \cos\left(\frac{\pi j}{4}\right) + 1 = 3 - 2 \cos\left(\frac{\pi j}{4}\right), \quad j = 1, 2, 3.$$

- (i) None of these are zero, so the matrix has an empty right null space so, by the rank-nullity theorem, is full rank: $r = 3$.

sim. seen ↓

1, A

- (ii) From the considerations above, the eigenvalues are

sim. seen ↓

$$2 - 2 \cos\left(\frac{\pi j}{4}\right) + 1 = 3 - 2 \cos\left(\frac{\pi j}{4}\right), \quad j = 1, 2, 3.$$

This answer is sufficient, but the numerical values are

$$3 - \sqrt{2}, \quad 3, \quad 3 + \sqrt{2}$$

3, A

- (b) (i) This matrix is circulant, therefore its eigenvectors (given on the exam handout) are

sim. seen ↓

$$\begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \omega^{3j} \end{pmatrix}, \quad \omega = e^{2\pi i/4} = i, \quad j = 0, 1, 2, 3.$$

There are linearly independent, so the matrix is full rank, i.e. the rank is 4. (Alternatively, this can be deduced in the same way as part (a) once the following 4 non-zero eigenvalues have been determined in part (ii)).

1, A

- (ii) To find the eigenvalues, simply multiply the eigenvectors by the matrix:

sim. seen ↓

$$\begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \omega^{3j} \end{pmatrix} = \lambda_j \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \omega^{3j} \end{pmatrix},$$

so the first element gives

$$\lambda_j = 1 + 4\omega^j + 3\omega^{2j} + 2\omega^{3j}, \quad j = 0, 1, 2, 3.$$

Therefore

$$\lambda_0 = 10, \quad \lambda_1 = 1 + 4i - 3 - 2i = -2 + 2i,$$

$$\lambda_2 = 1 - 4 + 3 - 2 = -2, \quad \lambda_3 = 1 - 4i - 3 + 2i = -2 - 2i,$$

In summary, the eigenvalues are

$$10, -2 + 2i, -2, -2 - 2i$$

4, A

- (c) (i) Since there is a single loop in the graph, we expect there to be a 1-parameter family of admissible currents in the minimization problem by Thomson's principle. By conservation of current at node $\boxed{+}$,

sim. seen $\Downarrow$

$$w_b = 1 - w_a$$

and

$$w_a = w_c = w_e, \quad w_b = w_d = w_f.$$

Therefore, the quantity to be minimized according to Thomson's principle is

$$\begin{aligned} \tilde{\mathcal{E}} &= \frac{w_a^2}{C} + \frac{w_c^2}{C} + \frac{w_e^2}{C} + \frac{w_b^2}{D} + \frac{w_d^2}{D} + \frac{w_f^2}{D} \\ &= \frac{3w_a^2}{C} + \frac{3w_b^2}{D} \\ &= \frac{3w_a^2}{C} + \frac{3(1-w_a)^2}{D}. \end{aligned}$$

For a minimum we require

$$\frac{\partial \tilde{\mathcal{E}}}{\partial w_a} = \frac{6w_a}{C} - \frac{6(1-w_a)}{D} = 0$$

or

$$w_a = \frac{C}{C+D}, \quad \text{so that} \quad w_b = \frac{D}{C+D}.$$

4, B

- (ii) The effective conductance is readily computed using the method of equivalent circuits and repeated use of the rules for the effective conductance of two conductors (of conductances c_a and c_b) in parallel,

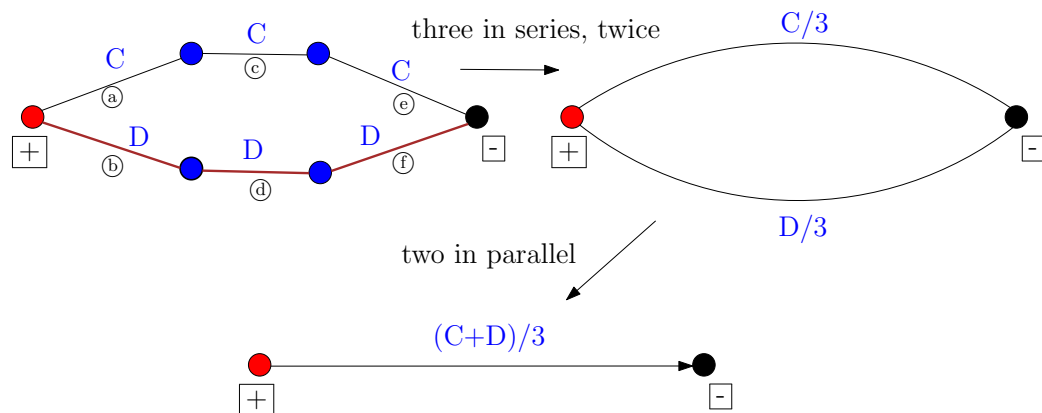
sim. seen $\Downarrow$

$$c_a + c_b$$

and in series,

$$\frac{c_a c_b}{c_a + c_b},$$

as shown in this Figure:



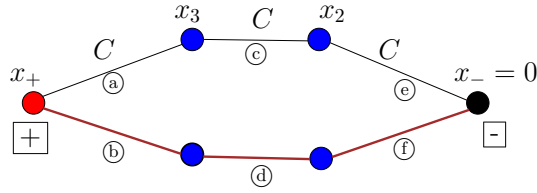
The effective conductance is therefore

$$\frac{C+D}{3}.$$

3, A

- (iii) The voltage x_+ at node $\boxed{+}$ for the solution in part (i) can be calculated using Ohm's law and the known currents. Let x_2, x_3 and x_+ be the voltages as shown in this figure:

sim. seen $\Downarrow$



Then, by Ohm's law the voltages at the upper blue nodes connected by edges of conductance C are

$$Cx_2 = w_e = \frac{C}{C+D}, \quad x_2 = \frac{1}{C+D}$$

and

$$C(x_3 - x_2) = w_c = \frac{C}{C+D}, \quad x_3 = \frac{2}{C+D}$$

so that

$$C(x_- - x_3) = w_a = \frac{C}{C+D}, \quad x_+ = \frac{3}{C+D}.$$

To make $x_+ = 1$ then all quantities, including the currents w_a and w_b , must be multiplied by $(C+D)/3$ leading to rescaled currents

$$\tilde{w}_a = \frac{(C+D)}{3}w_a = \frac{C}{3}, \quad \tilde{w}_b = \frac{(C+D)}{3}w_b = \frac{D}{3}$$

from which it follows that the effective conductance (defined as the total current divergence at node $\boxed{+}$ when $x_+ = 1$) is

$$\tilde{w}_a + \tilde{w}_b = \frac{C+D}{3}$$

which is consistent with part (ii).

4, B

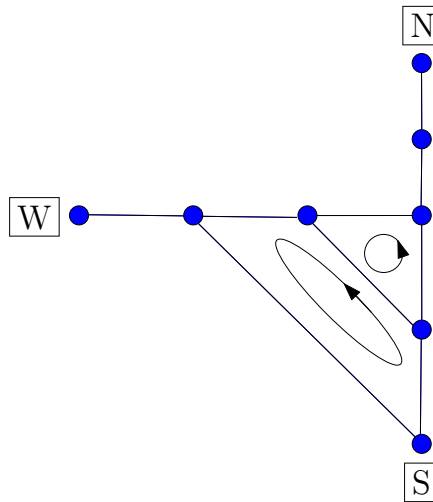
2. (a) (i) There are $m = 9$ edges and $n = 8$ nodes so any incidence matrix $\mathbf{A}$ is a 9-by-8 matrix.

sim. seen ↓

1, A

(ii) The graph is connected, so the dimension of the right null space of $\mathbf{A}$ is 1. There are two obvious independent, irreducible loops in the graph (as shown in the figure below) so the dimension of the left null space of $\mathbf{A}$ is 2. Alternatively, since the rank is $r = n - 1 = 8 - 1 = 7$ then, by rank-nullity, the dimension of the left null space is $m - r = 9 - 7 = 2$.

sim. seen ↓



(b) The question is asking for the escape probability p_{esc} from $\boxed{W}$ to $\boxed{S}$. By the analogy with electric circuits this is

2, A

sim. seen ↓

$$p_{\text{esc}} = C_{\text{eff}}, \quad (1)$$

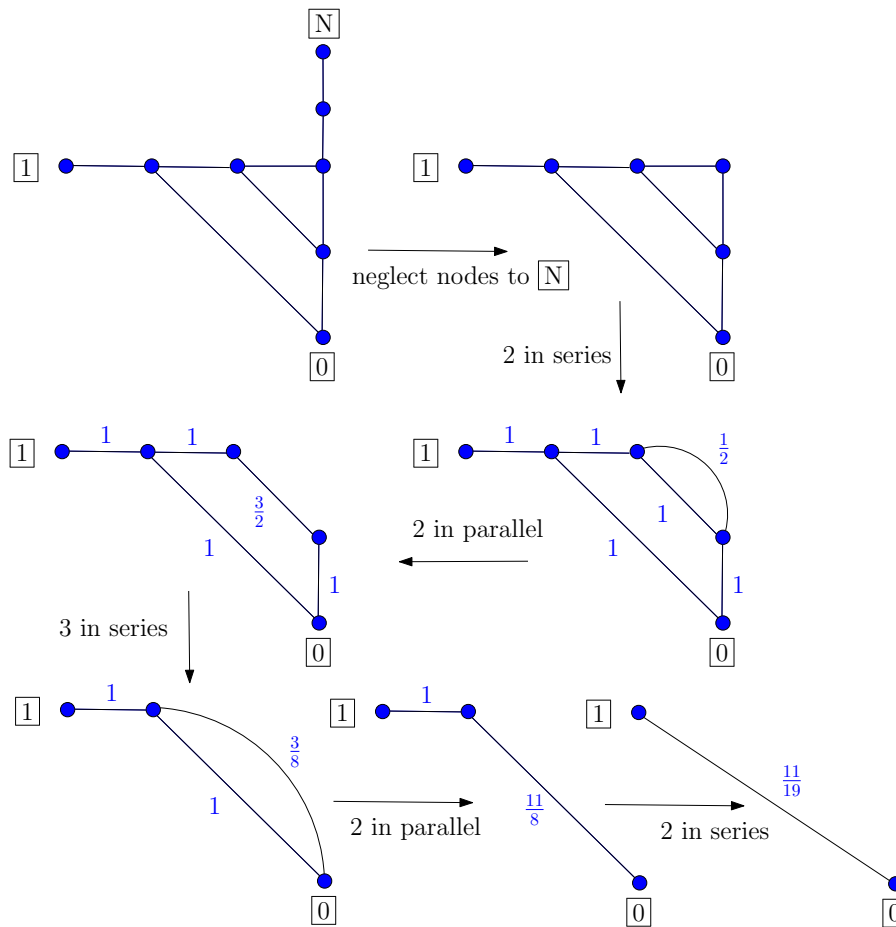
where C_{eff} is the effective conductance if all edges have unit conductance and node $\boxed{W}$ is set to unit voltage while node $\boxed{S}$ is grounded. The easiest way to compute C_{eff} is to use the “method of equivalent circuits” repeatedly using the rules for the effective conductance of two conductors (of conductances c_a and c_b) in parallel,

$$c_a + c_b$$

and in series,

$$\frac{c_a c_b}{c_a + c_b}.$$

The figure shows the sequence of reductions to equivalent circuits:



No current can flow through the two edges connecting the main graph to node $\boxed{N}$ by Kirchoff's current law and conservation of current, hence they can be ignored. The final result is

$$p_{\text{esc}} = C_{\text{eff}} = \frac{11}{19}.$$

8, B

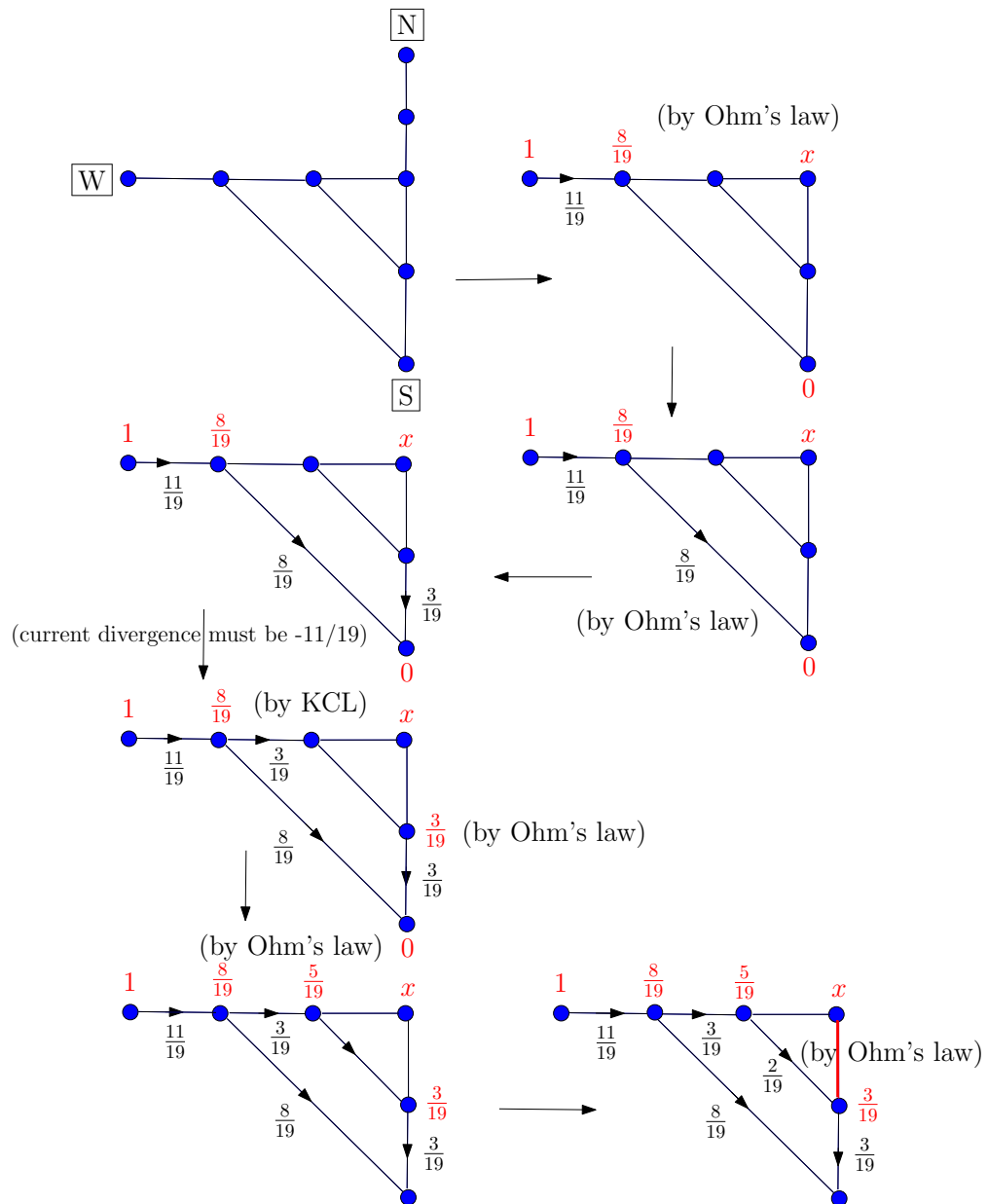
- (c) The question is asking for the hitting probability of node $\boxed{W}$ before node $\boxed{S}$ starting at node $\boxed{N}$. By the electric circuit analogy, this is the voltage at node $\boxed{N}$ is node $\boxed{W}$ is set to unit voltage and node $\boxed{S}$ is grounded. Note that since no current can flow in the two edges connecting the main graph to node $\boxed{N}$ then the voltage x (shown in red) at the node shown in the figure below is required.

sim. seen $\Downarrow$

Given the number of nodes and edges, the easiest way to find x is to use the result of part (b) on C_{eff} and to make repeated use of Ohm's law (that the current in an edge equals the voltage drop across it given that all edges have unit conductance) and KCL (that the net current into an given node equals the net current out of it, except at nodes $\boxed{W}$ and $\boxed{S}$). The figure below shows how successive use of these laws give the required x to be the solution of

$$x - \frac{3}{19} = \frac{1}{19}, \quad \text{or} \quad \boxed{x = \frac{4}{19}}$$

because, by KCL, the current in the edge (shown red) joining the node at voltage x to that at voltage $3/19$ that is connected to node $\boxed{S}$ must be $1/19$.



9, C

3. (a) The system to solve in equilibrium is

sim. seen ↓

$$\mathbf{K}\mathbf{x}_e = \mathbf{f}_e, \quad (2)$$

where

$$\mathbf{K} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{f}_e = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad (3)$$

where the reaction force at the wall must be -1 since this is required by $\mathbf{x}_0^T \mathbf{f} = 0$ where $\mathbf{x}_0 = (1, 1, 1)^T$ is the usual right null vector of $\mathbf{K}$. Since

$$\mathbf{x}_e = \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}, \quad (4)$$

it follows from the first equation that

$$x_1 = 1. \quad (5)$$

The 3rd equation then gives

$$x_2 = 2 \quad (6)$$

therefore

$$\mathbf{x}_e = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}. \quad (7)$$

- (b) The solution to part (a) provides that the reaction force at the wall is -1 since this is required by $\mathbf{x}_0^T \mathbf{f} = 0$ where $\mathbf{x}_0 = (1, 1, 1)^T$ is the usual right null vector of $\mathbf{K}$. Had this requirement not been used in part (a), and the second and third equations solved for x_1 and x_2 , then the first equation would have furnished this result.

4, A

sim. seen ↓

2, A

- (c) The system to be solved when the spring-mass system is out of equilibrium is

sim. seen ↓

$$\mathbf{f} + \mathbf{f}_I = \mathbf{M} \frac{d^2 \mathbf{x}}{dt^2}, \quad \mathbf{x} = \mathbf{x}_e + \hat{\mathbf{x}}, \quad (8)$$

where $\mathbf{M}$ is the diagonal mass matrix (the 2-by-2 identity here) and

$$\mathbf{f}_I = -\mathbf{K}\mathbf{x}, \quad \mathbf{f} = \mathbf{f}_e + \hat{\mathbf{f}}, \quad \hat{\mathbf{f}} = \begin{pmatrix} R(t) \\ 0 \\ 0 \end{pmatrix}, \quad (9)$$

where $R(t)$ is the reaction force at the ceiling in the unsteady case. Therefore,

$$\mathbf{f}_e + \hat{\mathbf{f}} - \mathbf{K}(\mathbf{x}_e + \hat{\mathbf{x}}) = \frac{d^2 \hat{\mathbf{x}}}{dt^2} \quad (10)$$

which reduces to

$$\hat{\mathbf{f}} - \mathbf{K}\hat{\mathbf{x}} = \frac{d^2 \hat{\mathbf{x}}}{dt^2}. \quad (11)$$

The second and third components of this matrix equation gives the standard eigenvalue problem

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \Phi_0 = \lambda \Phi_0 \quad (12)$$

where

$$\hat{\mathbf{x}} = \begin{pmatrix} 0 \\ \Phi(t) \end{pmatrix}, \quad \Phi(t) = \Phi_0 e^{i\omega t}, \quad \lambda = \omega^2 \quad (13)$$

The determinant condition is

$$\det \begin{pmatrix} 2 - \lambda & -1 \\ -1 & 1 - \lambda \end{pmatrix} = 0 \quad (14)$$

or

$$(2 - \lambda)(1 - \lambda) - 1 = 0, \quad \lambda^2 - 3\lambda + 1 = 0 \quad (15)$$

giving

$$\lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}. \quad (16)$$

Therefore the two frequencies of oscillation are

$$\omega_{\pm} = \sqrt{\frac{3 \pm \sqrt{5}}{2}}. \quad (17)$$

6, A

- (d) The corresponding eigenvectors from the calculation in part (c) are easily found to be

sim. seen $\Downarrow$

$$\lambda_+, \quad \begin{pmatrix} 1 \\ 2 - \lambda_+ \end{pmatrix} \quad (18)$$

and

$$\lambda_-, \quad \begin{pmatrix} 1 \\ 2 - \lambda_- \end{pmatrix} \quad (19)$$

Therefore

$$\begin{aligned} \hat{\mathbf{x}}(t) = & (c_+ e^{i\omega_+ t} + \overline{c_+} e^{-i\omega_+ t}) \begin{pmatrix} 1 \\ 2 - \lambda_+ \end{pmatrix} \\ & + (c_- e^{i\omega_- t} + \overline{c_-} e^{-i\omega_- t}) \begin{pmatrix} 1 \\ 2 - \lambda_- \end{pmatrix} \end{aligned} \quad (20)$$

for some complex constants $c_{\pm}$. Imposing the zero initial velocity condition implies that

$$\hat{\mathbf{x}}(t) = A \cos(\omega_+ t) \begin{pmatrix} 1 \\ 2 - \lambda_+ \end{pmatrix} + B \cos(\omega_- t) \begin{pmatrix} 1 \\ 2 - \lambda_- \end{pmatrix} \quad (21)$$

for some real constants A and B . The initial displacement conditions require that

$$A + B = 1, \quad A(2 - \lambda_+) + B(2 - \lambda_-) = 0. \quad (22)$$

Solving these gives

$$A = \frac{2 - \lambda_-}{\lambda_+ - \lambda_-}, \quad B = -\frac{2 - \lambda_+}{\lambda_+ - \lambda_-}. \quad (23)$$

The required solution is then

$$\hat{\mathbf{x}}(t) = \frac{(2 - \omega_-^2) \cos(\omega_+ t)}{\omega_+^2 - \omega_-^2} \begin{pmatrix} 1 \\ 2 - \omega_+^2 \end{pmatrix} - \frac{(2 - \omega_+^2) \cos(\omega_- t)}{\omega_+^2 - \omega_-^2} \begin{pmatrix} 1 \\ 2 - \omega_-^2 \end{pmatrix}.$$

8, D

4. (a) From the exam handout, the ordinary differential equation governing the voltage potential $\phi(x)$ in each section of the conductor is

sim. seen ↓

$$-\frac{d}{dx} \left(C(x) \frac{d\phi}{dx} \right) = F(x),$$

where $C(x)$ is the conductivity function and $F(x)$ is the current source density function. Given that the conductivity is unity, the equations for $\phi_{\pm}(x)$ are

$$\begin{aligned} -\frac{d^2\phi_-}{dx^2} &= -f_0, & 0 < x < 1, \\ -\frac{d^2\phi_+}{dx^2} &= +f_0 & 1 < x < 2, \end{aligned}$$

where the expressions for the current source density function given in the question have been used.

The boundary conditions are $\phi_-(0) = 0$ and $\phi_+(2) = V$ and the voltage potentials and currents must be continuous at $x = 1$:

$$\phi_-(1) = \phi_+(1), \quad -\frac{d\phi_-}{dx} \Big|_{x=1} = -\frac{d\phi_+}{dx} \Big|_{x=1}$$

- (b) The solutions, satisfying $\phi_-(0) = 0$ and $\phi_+(2) = V$, are

3, A

sim. seen ↓

$$\phi(x) = \begin{cases} \frac{f_0 x^2}{2} + Ax, & 0 < x < 1, \\ -\frac{f_0 x^2}{2} + Bx + (V - 2B + 2f_0), & 1 < x < 2, \end{cases}$$

where A and B are to be determined. These follow by enforcing that the voltage potentials and currents are continuous at $x = 1$. The condition $\phi_-(1) = \phi_+(1)$ implies

$$\frac{f_0}{2} + A = -\frac{f_0}{2} + B + (V - 2B + 2f_0), \quad \text{or} \quad A + B = V + f_0. \quad (24)$$

The condition that the currents are continuous requires

$$\frac{d\phi_-}{dx} \Big|_{x=1} = \frac{d\phi_+}{dx} \Big|_{x=1}$$

which gives

$$f_0 + A = -f_0 + B, \quad \text{or} \quad A - B = -2f_0. \quad (25)$$

Solving (24) and (25) gives

$$A = \frac{V - f_0}{2}, \quad B = \frac{V + 3f_0}{2}.$$

The solution is therefore

$$\phi(x) = \begin{cases} \frac{f_0 x^2}{2} + \left(\frac{V - f_0}{2} \right) x, & 0 < x < 1, \\ -\frac{f_0 x^2}{2} + \left(\frac{V + 3f_0}{2} \right) x - f_0, & 1 < x < 2. \end{cases}$$

8, D

- (c) If there is to be no current at $x = 0$ then

sim. seen ↓

$$\left. \frac{d\phi_-}{dx} \right|_{x=0} = 0$$

implying that

$$V = f_0.$$

3, C

- (d) With $V = f_0$ the solution from part (b) reduces to

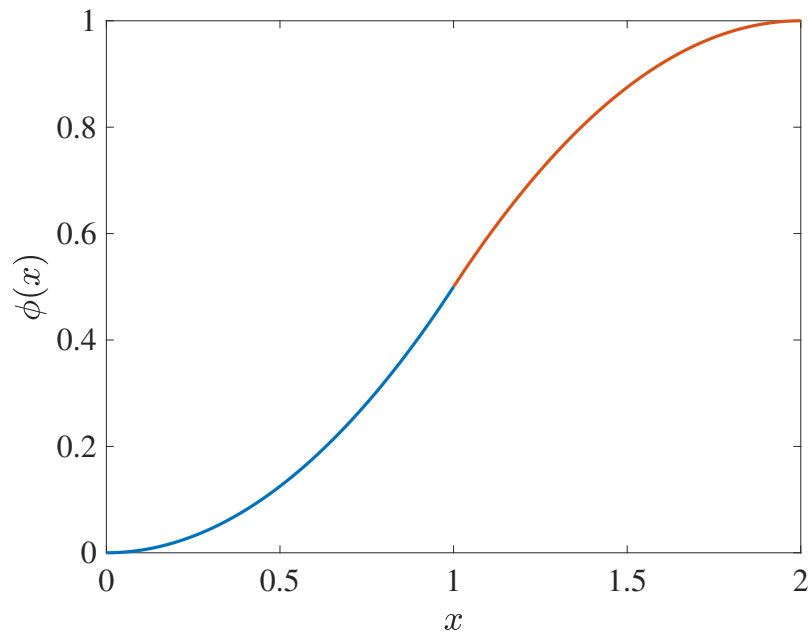
sim. seen ↓

$$\phi(x) = \begin{cases} \frac{f_0 x^2}{2}, & 0 < x < 1, \\ -\frac{f_0 x^2}{2} + 2f_0 x - f_0, & 1 < x < 2. \end{cases}$$

2, A

- (e) A sketch of this voltage distribution for $f_0 = 1$ is given in the figure:

sim. seen ↓



4, B

Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total marks: 80 of 80 marks

Total Mastery marks: 0 of 20 marks

MATH40007 An Introduction to Applied Mathematics Markers Comments

- Question 1 This question was a good test in that only a dozen or so students obtained full marks, but just as few scoring very poorly. There was a good spread of marks.
Part (c) was found most difficult, especially part (c)(iii), where few students gave clearly explained answers.
The students used a variety of methods from linear algebra to find the eigenvalues in parts (a) and (b), even though all this could be avoided if the structure of the matrices was examined more carefully (as was intended). It should be emphasized that full marks were awarded for correct answers no matter what method was used.
- Question 2 This question was well answered, with a good percentage of students scoring full marks.
Parts (b) and (c) were answered better than the examiner expected: they are (quite) heavy on algebra/manipulations, but this did not impede the students who performed impressively here. Algebraic slips were penalized gently, although students who neglected to explain their solution strategy lost significant points, especially if the final answer was also incorrect.
- Question 3 Parts (a) and (b) were answered well with most students scoring full marks. A significant number of students did not notice that many of the constants could be set to one or that no gravity was active. Full marks were still awarded if these simplifications were not made, however, this made the question much trickier than intended.
A significant number of students did not attempt part (c). Those that did, generally recognised that the frequencies were governed by an eigenvalue problem and almost all students who found the eigenvalue problem solved it correctly. Marks were mostly lost due to students not explaining where the eigenvalue problem came from, ignoring the external forcing and equilibrium position. A considerable number of students considered the full Laplacian instead of the reduced Laplacian which resulted in them finding the wrong frequencies. Most students did not attempt part (d). Those who did attempt it struggled to recall the correct general solution, making it difficult to recover the desired result.
- Question 4 The distribution of marks for this question was quite big due to the fact that many students didn't attempt parts c, d and e and from those who attempted all parts, only a few achieved high or full marks. Most students didn't include the boundary condition arising from the continuity of current and thus the following parts were answered incorrectly as well. Only a few students included this boundary condition and thus were able to achieve full marks for this question.