

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
Summer 2026

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

An Introduction to Applied Mathematics

Date: Friday, 08 May, 2026

Time: Start time 14:00 – End time 16:00 (BST)

Time Allowed: 2 hours

This paper has 4 Questions.

Please Answer Each Question in a Separate Answer Booklet

This is a closed book examination.

Candidates should start their solutions to each question on a new sheet of paper.

Supplementary books may only be used after the relevant main book(s) are full.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Allow margins for marking.

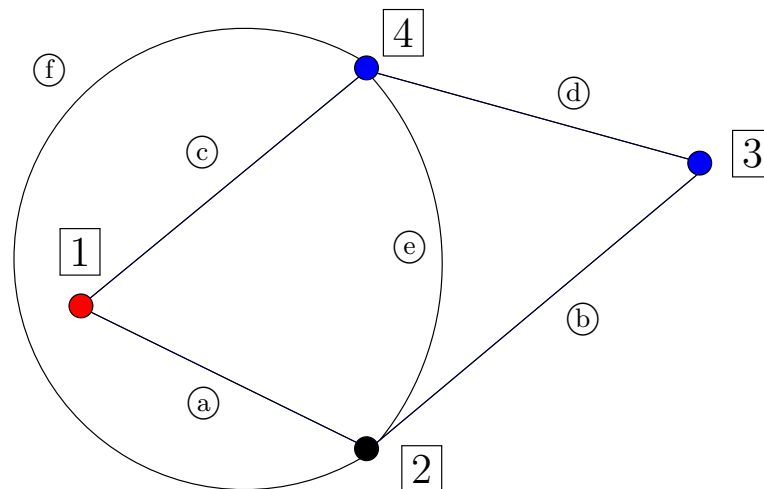
DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO DO SO

1. (a) Consider the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ 3 & 6 & -6 \\ 4 & 8 & -8 \end{pmatrix}.$$

- (i) Find the column space of $\mathbf{M}$. (1 mark)
- (ii) Find the right null space of $\mathbf{M}$. (3 marks)
- (iii) Find the row space of $\mathbf{M}$. (1 mark)
- (iv) Find the left null space of $\mathbf{M}$. (3 marks)

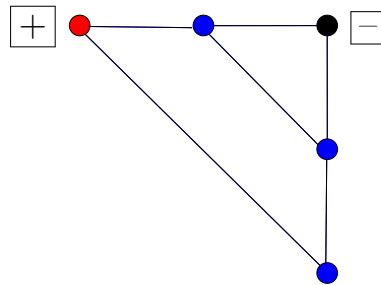
(b) The figure shows an electric circuit where each edge is a conductor of unit conductance. Node 1 is set to unit voltage, node 2 is grounded, and Kirchhoff's current law holds at all other nodes.



- (i) Find the voltage at node 4. (4 marks)
- (ii) Find the current in edge (d) and determine its direction. (4 marks)
- (iii) Find the effective conductance of the circuit. (4 marks)

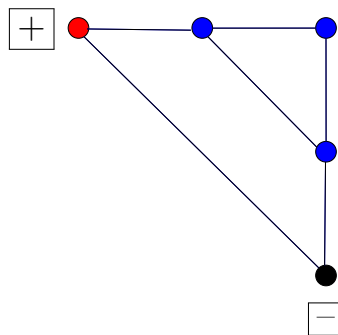
(Total: 20 marks)

2. (a) Find the effective conductance of the circuit shown below where all edges are conductors of unit conductance and where node $\boxed{+}$ is set to unit voltage and node $\boxed{-}$ is grounded.



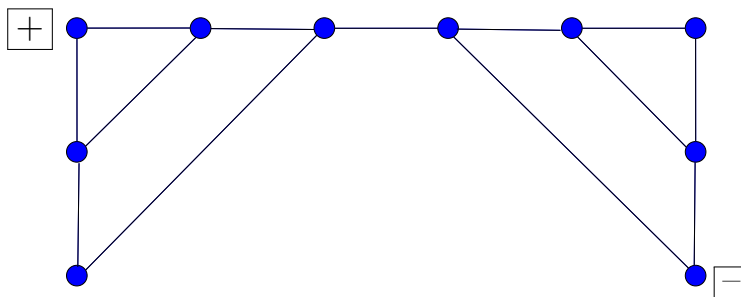
(8 marks)

- (b) Find the effective conductance of the circuit shown below where all edges are conductors of unit conductance and where node $\boxed{+}$ is set to unit voltage and node $\boxed{-}$ is grounded.



(6 marks)

- (c) A random walker takes a journey on the graph shown below. At each node, the walker makes a random choice as to which connected edge to traverse next, with each edge having equal probability of being chosen. The journey stops if the walker reaches node $\boxed{+}$ or node $\boxed{-}$. On a journey starting from node $\boxed{+}$, find the probability of reaching node $\boxed{-}$ before returning to node $\boxed{+}$.



(6 marks)

(Total: 20 marks)

3. This question concerns unsteady particle diffusion on the 4-node linear graph shown in the figure. Let $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$ be the node vector of particle concentrations at the 4 nodes.



The vector of particle fluxes along the edges, denoted by $\mathbf{w}$, is given by $\mathbf{w} = -\mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is an incidence matrix for the graph. The evolution in time of the node potential vector $\mathbf{x}(t)$ is given by

$$\frac{d\mathbf{x}}{dt} = -\mathbf{f}_I + \mathbf{f},$$

where $\mathbf{f}_I$ is the vector of divergences of the particle fluxes $\mathbf{w}$ at the nodes, and $\mathbf{f}$ represents external sources of particles at the nodes.

Consider the problem where the concentration at node 4 is fixed to be zero and the concentration at node 1 is fixed to be 1. There are no external sources of particles at nodes 2 and 3. The initial node concentrations, at $t = 0$, are given by the 4-vector $(1, 1, 1, 0)^T$. The solution $\mathbf{x}(t)$ for the node concentrations therefore has the form

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ \hat{\mathbf{x}}(t) \\ 0 \end{pmatrix}, \quad \hat{\mathbf{x}}(t) = \begin{pmatrix} x_2(t) \\ x_3(t) \end{pmatrix}, \quad \hat{\mathbf{x}}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- (a) Show that the solution for $\hat{\mathbf{x}}(t)$ can be decomposed as

$$\hat{\mathbf{x}}(t) = \mathbf{x}_s + \tilde{\mathbf{x}}(t),$$

where the constant vector $\mathbf{x}_s$ and the time-dependent vector $\tilde{\mathbf{x}}(t)$ satisfy

$$\mathbf{K}_2 \mathbf{x}_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{d\tilde{\mathbf{x}}(t)}{dt} = -\mathbf{K}_2 \tilde{\mathbf{x}}(t),$$

for a suitable matrix $\mathbf{K}_2$ which you should write down explicitly. (4 marks)

- (b) By writing

$$\mathbf{x}_s = a_1 \Phi_1 + a_2 \Phi_2,$$

where Φ_1 and Φ_2 are the real orthonormal eigenvectors of $\mathbf{K}_2$, find the two constants a_1 and a_2 and the vector $\mathbf{x}_s$. (4 marks)

- (c) By writing

$$\tilde{\mathbf{x}}(t) = b_1(t) \Phi_1 + b_2(t) \Phi_2$$

for some time-dependent functions $b_1(t)$ and $b_2(t)$, determine $\tilde{\mathbf{x}}(t)$ and, by combining it with the result of part (b), find the solution for $\hat{\mathbf{x}}(t)$. (8 marks)

- (d) On the same axes, sketch graphs of $x_2(t)$ and $x_3(t)$ as functions of time for $t > 0$.

(4 marks)

(Total: 20 marks)

4. A metal rod of uniform cross-sectional area occupies the interval $0 \leq x \leq 1$ of the x -axis. It is a conductor both of current and heat. The steady conduction of heat is governed by Fourier's law which is analogous to Ohm's law for the conduction of current: for heat conduction, the temperature $T(x)$ plays the role of the voltage $\phi(x)$, heat flux plays the role of the current, and thermal conductivity plays the role of electrical conductivity. This rod has uniform electrical conductivity equal to unity and uniform thermal conductivity also equal to unity.

(a) The end of the rod at $x = 1$ is held at unit voltage and the end at $x = 0$ is grounded. Assuming there are no sources of current for $0 < x < 1$, write down the differential equation and boundary conditions satisfied by the steady-state voltage $\phi(x)$. (2 marks)

(b) The electrical energy dissipation density at each location $0 < x < 1$ in the interior of the rod is given by

$$-\frac{d\phi}{dx} \times J(x),$$

where $J(x)$ is the current density. The two ends of the rod at $x = 0$ and $x = 1$ are held at unit temperature. Write down the differential equation and boundary conditions satisfied by the temperature $T(x)$ assuming that the electrical energy dissipation density at each point x in the interior of the rod provides a heat source density at that point and that all the dissipated electrical energy is converted into heat energy.

(4 marks)

(c) Find the solutions for $\phi(x)$ and $T(x)$.

(6 marks)

(d) Calculate heat flux leaving the rod at each of its ends.

(4 marks)

(e) The device attached to the end of the rod at $x = 0$ keeping it at unit temperature is now removed so that there is no external heat sink/source at $x = 0$. The end of the rod at $x = 1$ is, however, still kept at unit temperature. Find the temperature distribution $T(x)$ in this new situation.

(3 marks)

(f) Calculate heat flux leaving the rod at $x = 1$ for the situation in part (e).

(1 mark)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2026

This paper is also taken for the relevant examination for the Associateship.

M40007

M40007 (Solutions)

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1. (a) (i) It is evident that the second and third columns of $\mathbf{M}$ are multiples of the first column $\mathbf{c}_1$, i.e, $\mathbf{c}_2 = 2\mathbf{c}_1$ and $\mathbf{c}_3 = -2\mathbf{c}_1$. Therefore the column space is the space spanned by

meth seen ↓

1, A

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\rangle \quad (1)$$

or, equivalently, all vectors of the form

$$a \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad (2)$$

where a is a constant.

meth seen ↓

- (ii) Since

$$2\mathbf{c}_1 - \mathbf{c}_2 = 0, \quad 2\mathbf{c}_1 + \mathbf{c}_3 = 0 \quad (3)$$

3, A

then the two right null vectors are

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \quad (4)$$

The right null space is therefore the space spanned by

$$\left\langle \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad (5)$$

or equivalently any vector of the form

$$a_1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad (6)$$

where a_1 and a_2 are constants.

meth seen ↓

- (iii) The row space of $\mathbf{M}$ is the column space of the matrix

1, A

$$\mathbf{M}^T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -2 & -4 & -6 & -8 \end{pmatrix}. \quad (7)$$

This is evidently the space spanned by

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right\rangle \quad (8)$$

or, equivalently, all vectors of the form

$$a \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad (9)$$

where a is a constant.

(iv) Letting the columns of $\mathbf{M}^T$ be $\hat{\mathbf{c}}_j, j = 1, 2, 3, 4$ then evidently

$$\hat{\mathbf{c}}_2 = 2\hat{\mathbf{c}}_1, \quad \hat{\mathbf{c}}_3 = 3\hat{\mathbf{c}}_1, \quad \hat{\mathbf{c}}_4 = 4\hat{\mathbf{c}}_1, \quad (10)$$

or

$$2\hat{\mathbf{c}}_1 - \hat{\mathbf{c}}_2 = 0, \quad 3\hat{\mathbf{c}}_1 - \hat{\mathbf{c}}_3 = 0, \quad 4\hat{\mathbf{c}}_1 - \hat{\mathbf{c}}_4 = 0 \quad (11)$$

so that the following 3 vectors are in the right null space of $\mathbf{M}^T$ (which is the left null space of $\mathbf{M}$):

$$\hat{\mathbf{x}}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\mathbf{x}}_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \hat{\mathbf{x}}_3 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \quad (12)$$

The left null space of $\mathbf{M}$ is therefore the space spanned by

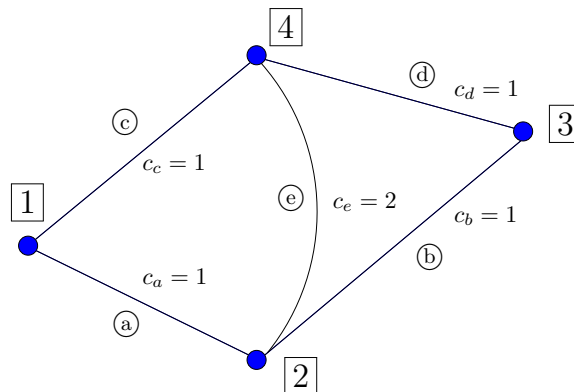
$$\left\langle \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\rangle \quad (13)$$

or equivalently any vector of the form

$$a_1 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 4 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad (14)$$

where a_1, a_2 and a_3 are constants.

- (b) Using the rule for two conductors in parallel, where the conductances add, the given circuit is equivalent to



Let the voltage vector be $\mathbf{x}$ and the current divergence vector be $\mathbf{f}$. Then these are related by

$$\mathbf{K}\mathbf{x} = \mathbf{f}, \quad (15)$$

where

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ x_3 \\ x_4 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} C_{\text{eff}} \\ -C_{\text{eff}} \\ 0 \\ 0 \end{pmatrix}, \quad (16)$$

and

$$\mathbf{K} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -1 & -2 \\ 0 & -1 & 2 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}. \quad (17)$$

The subsystem to solve is

$$\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (18)$$

The solution is

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/7 \\ 2/7 \end{pmatrix}. \quad (19)$$

- (i) The voltage at node 4 is therefore

$$x_4 = \frac{2}{7} \quad (20)$$

- (ii) By Ohm's law, using the unit conductance value, the current in edge d is $x_4 - x_3$ or

$$w_d = \frac{2}{7} - \frac{1}{7} = \frac{1}{7} \quad (21)$$

It is directed along the voltage drop from node 4 to node 3.

- (iii) By Ohm's law, using the unit conductance values, the current w_a in edge a is +1 and the current w_c in edge c is $1 - 2/7 = 5/7$ so the total current divergence at node 1, which is the effective conductance, is $w_a + w_c$ or

$$C_{\text{eff}} = 1 + \frac{5}{7} = \frac{12}{7} \quad (22)$$

Note: Another approach is to calculate C_{eff} using the method of equivalent circuits and then the voltage x_4 and the current in edge c can be computed using Ohm's law.

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4, B

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4, B

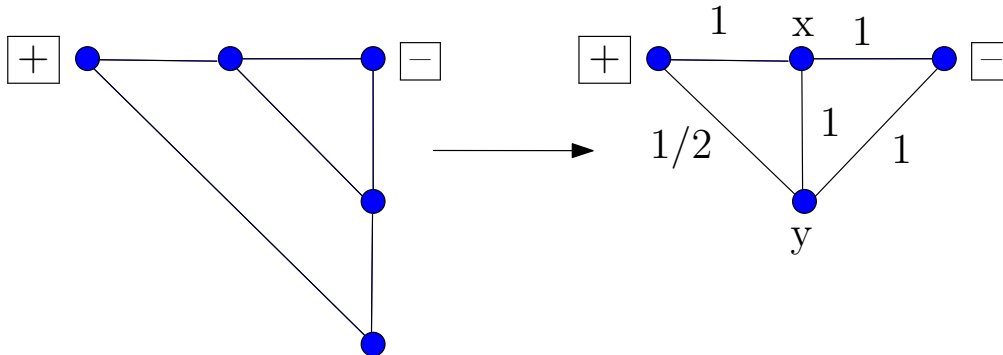
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4, B

2. (a) By the method of equivalent circuits, two of the edges are in series and can be replaced by a single conductor of conductance $1/2$ as shown in the figure. Then assign x and y as the voltages at the two remaining nodes where KCL holds.

meth seen ↓

8, C



These can be found by imposing the averaging condition at these nodes where KCL holds. These take the form

$$(x - 1) + (x - y) + x = 0, \quad (23)$$

$$\frac{1}{2}(y - 1) + (y - x) + y = 0. \quad (24)$$

These are

$$3x - y = 1 \quad (25)$$

$$-x + \frac{5y}{2} = \frac{1}{2} \implies -3x + \frac{15y}{2} = \frac{3}{2}. \quad (26)$$

Addition of these two equations gives

$$\frac{13y}{2} = \frac{5}{2}, \quad y = \frac{5}{13}. \quad (27)$$

Hence

$$x = \frac{5y}{2} - \frac{1}{2} = \frac{25}{26} - \frac{13}{26} = \frac{12}{26} = \frac{6}{13}. \quad (28)$$

The effective conductance is the current out of node $\boxed{+}$:

$$(1 - x) + \frac{1}{2}(1 - y) = 1 - \frac{6}{13} + \frac{1}{2} \left(1 - \frac{5}{13} \right) = \frac{7}{13} + \frac{4}{13} = \frac{11}{13}. \quad (29)$$

meth seen ↓

- (b) This circuit has a symmetry: by subtracting off unity from all node potentials, then reversing all currents, the effective conductance of the resulting circuit is the same as that sought, as it evident from the figure:

6, D

Using KCL at the two nodes with potential x and y , respectively, gives

$$x + (x - (1 - x)) + x - y = 0 \quad (30)$$

and

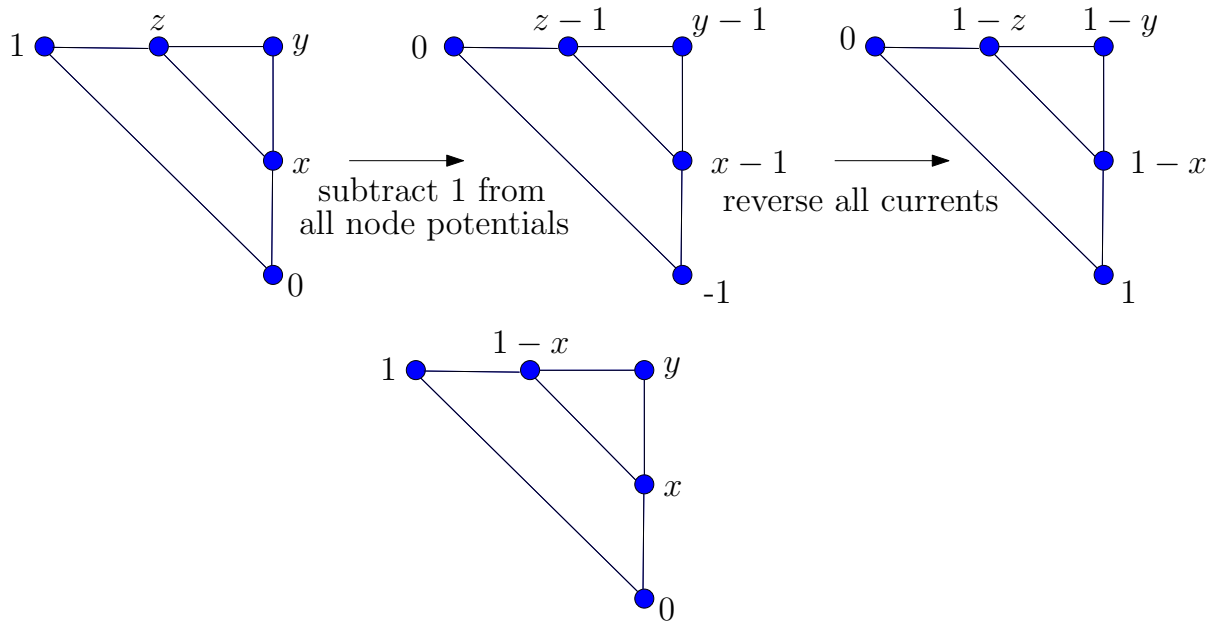
$$(y - (1 - x)) + (y - x) = 0. \quad (31)$$

These are

$$4x - y = 1 \quad (32)$$

and

$$2y = 1 \implies y = \frac{1}{2}. \quad (33)$$



Hence

$$4x = 1 + y = \frac{3}{2} \implies x = \frac{3}{8} \quad (34)$$

The effective conductance is the total current out of node $\boxed{+}$ which is

$$1 + (1 - (1 - x)) = 1 + x = \frac{11}{8}. \quad (35)$$

- (c) We are seeking the "escape probability" p_{esc} from node $\boxed{+}$ to $\boxed{-}$. By the electric circuit analogy, and since there are two edges attached to node $\boxed{+}$, this is

meth seen $\Downarrow$

6, A

$$p_{\text{esc}} = \frac{C_{\text{eff}}}{2}, \quad (36)$$

where C_{eff} is the effective conductance if node $\boxed{+}$ is fixed to unit voltage and node $\boxed{-}$ grounded. But this is the effective conductance of parts (a) and (b) in series with a unit-conductance conductor joining them, meaning that

$$\frac{1}{C_{\text{eff}}} = 1 + \frac{13}{11} + \frac{8}{11} = \frac{32}{11}. \quad (37)$$

Therefore,

$$p_{\text{esc}} = \frac{C_{\text{eff}}}{2} = \frac{11}{64}. \quad (38)$$

3. (a) The governing equation is

meth seen ↓

$$\frac{d\mathbf{x}(t)}{dt} = -\mathbb{K}_4\mathbf{x}(t) + \mathbf{f}(t), \quad (39)$$

4, C

where $\mathbb{K}_4$ is the Laplacian of a 4-node linear graph. More explicitly, this equation is

$$\begin{pmatrix} 0 \\ dx_2/dt \\ dx_3/dt \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} f_1(t) \\ 0 \\ 0 \\ f_4(t) \end{pmatrix}. \quad (40)$$

The first and last equation give

$$x_2 - 1 + f_1(t) = 0, \quad x_3 + f_4(t) = 0 \quad (41)$$

which determine $f_1(t)$ and $f_4(t)$ once $x_2(t)$ and $x_3(t)$ have been found. The latter are determined by the middle two equations which have the form

$$\frac{d\hat{\mathbf{x}}(t)}{dt} = -\mathbf{K}_2\hat{\mathbf{x}}(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (42)$$

where

$$\mathbf{K}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \quad (43)$$

If

$$\hat{\mathbf{x}}(t) = \mathbf{x}_s + \tilde{\mathbf{x}}(t), \quad (44)$$

then on substitution into (42), it is clear that

$$\frac{d\tilde{\mathbf{x}}(t)}{dt} = -\mathbf{K}_2(\mathbf{x}_s + \tilde{\mathbf{x}}(t)) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (45)$$

which is solved if

$$\mathbf{K}_2\mathbf{x}_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (46)$$

and

$$\frac{d\tilde{\mathbf{x}}(t)}{dt} = -\mathbf{K}_2\tilde{\mathbf{x}}(t). \quad (47)$$

meth seen ↓

(b) The matrix $\mathbf{K}_2$ is circulant so its eigenvectors are

4, A

$$\begin{pmatrix} 1 \\ \omega^k \end{pmatrix}, \quad \omega = e^{2\pi i/2} = -1, \quad k = 0, 1. \quad (48)$$

(This result can be quoted from the exam handout, or it is easy to work out the eigenvectors directly). This gives orthonormal eigenvectors

$$\Phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_1 = 1; \quad \Phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = 3. \quad (49)$$

On substitution of the form

$$\mathbf{x}_s = a_1\Phi_1 + a_2\Phi_2 \quad (50)$$

into the governing steady equation,

$$\mathbf{K}_2(a_1\Phi_1 + a_2\Phi_2) = a_1\Phi_1 + 3a_2\Phi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (51)$$

Use of orthonormality leads to

$$a_1 = \frac{1}{\sqrt{2}}, \quad a_3 = \frac{1}{3\sqrt{2}} \quad (52)$$

hence

$$\mathbf{x}_s = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}. \quad (53)$$

(c) The next step is to write

$$\hat{\mathbf{x}}(t) = b_1(t)\Phi_1 + b_2(t)\Phi_2 \quad (54)$$

and to substitute this into (42):

$$\frac{db_1(t)}{dt}\Phi_1 + \frac{db_2(t)}{dt}\Phi_2 = -\mathbf{K}_2(b_1(t)\Phi_1 + b_2(t)\Phi_2) = -b_1(t)\Phi_1 - 3b_2(t)\Phi_2, \quad (55)$$

where the properties of the eigenvectors have been used. The orthonormality of those eigenvectors now implies

$$\frac{db_1(t)}{dt} = -b_1(t), \quad \frac{db_2(t)}{dt} = -3b_2(t). \quad (56)$$

These ordinary differential equations have solutions

$$b_1(t) = b_1(0)e^{-t}, \quad b_2(t) = b_2(0)e^{-3t}. \quad (57)$$

It follows from the initial conditions that

$$\hat{\mathbf{x}}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} + \tilde{\mathbf{x}}(0) \quad (58)$$

implying

$$\tilde{\mathbf{x}}(0) = b_1(0)\Phi_1 + b_2(0)\Phi_2 = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}. \quad (59)$$

The orthonormality of the eigenvectors now implies

$$b_1(0) = \frac{1}{\sqrt{2}}, \quad b_2(0) = -\frac{1}{3\sqrt{2}}. \quad (60)$$

Hence,

$$\tilde{\mathbf{x}}(t) = b_1(0)e^{-t}\Phi_1 + b_2(0)e^{-3t}\Phi_2 = \frac{e^{-t}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{e^{-3t}}{6} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (61)$$

and therefore

$$\hat{\mathbf{x}}(t) = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} + \frac{e^{-t}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{e^{-3t}}{6} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (62)$$

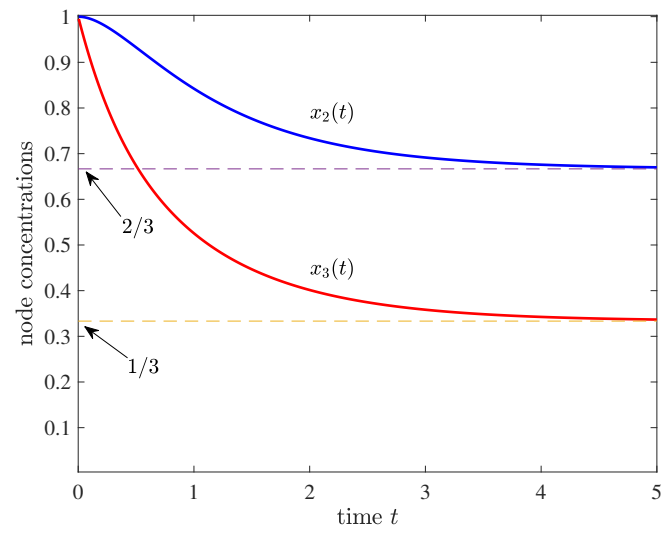
(d) The graph of the node concentrations as functions of time is shown in the figure. A sketch is sufficient showing the key features which are that, as expected, $\hat{\mathbf{x}}(t) \rightarrow \mathbf{x}_s$ as $t \rightarrow \infty$ since this is the steady solution when one boundary is held at unit concentration and the other grounded.

meth seen ↓

8, B

meth seen ↓

4, A



4. (a) Following lecture notes, or what is provided on the exam handout, the current $W(x)$ is given by

meth seen ↓

$$W(x) = -\frac{d\phi}{dx} \quad (63)$$

2, A

since the electrical conductivity is unity and, if there are no current sources, then

$$\frac{d}{dx}W(x) = 0. \quad (64)$$

It follows that

$$-\frac{d^2\phi}{dx^2} = 0. \quad (65)$$

The boundary conditions are

$$\phi(0) = 0, \quad \phi(1) = 1. \quad (66)$$

unseen ↓

- (b) Since the energy dissipation is a local source of heat, and that the thermal conductivity is unity, then, following the analogy with electric circuits explained in the question, the equation for $T(x)$ is

4, A

$$-\frac{d}{dx}\left(\frac{dT}{dx}\right) = \frac{dW}{dx}\left(-\frac{d\phi}{dx}\right) = W(x)^2, \quad (67)$$

since the thermal conductivity is unity and where the right hand side is a heat source term. Since both ends of the rod are held at unit temperature the boundary conditions are

$$T(0) = 1, \quad T(1) = 1. \quad (68)$$

unseen ↓

- (c) The boundary value problem for ϕ in part (a) has solution

6, D

$$\phi = x \quad (69)$$

which satisfies both boundary conditions. Since $dW/dx = 1$ the differential equation for $T(x)$ is therefore

$$-\frac{d}{dx}\left(\frac{dT}{dx}\right) = 1. \quad (70)$$

The general solution is easily found to be

$$T = -\frac{x^2}{2} + Ax + B \quad (71)$$

Using the boundary conditions to determine A and B leads to

$$T = 1 + \frac{x(1-x)}{2}. \quad (72)$$

meth seen ↓

- (d) The heat flux is

$$-\frac{dT}{dx} = -\frac{1}{2}(1-2x). \quad (73)$$

4, A

At $x = 0$ this is

$$-\frac{dT}{dx}\bigg|_{x=0} = -\frac{1}{2} \quad (74)$$

and at $x = 1$ this is

$$-\frac{dT}{dx}\bigg|_{x=1} = \frac{1}{2} \quad (75)$$

The heat flux *leaving* the rod at $x = 0$ is therefore

$$-\left(-\frac{dT}{dx}\right)\Big|_{x=0} = \frac{1}{2} \quad (76)$$

and that leaving the rod at $x = 1$ is

$$-\frac{dT}{dx}\Big|_{x=1} = \frac{1}{2}. \quad (77)$$

[Note: the total electrical energy dissipation in the rod is

$$\int_0^1 \left(\frac{dW}{dx}\right)^2 dx = 1 \quad (78)$$

and this is conducted away from the rod as heat equally at each end.]

- (e) The differential equation for $T(x)$ is still

$$-\frac{d}{dx} \left(\frac{dT}{dx} \right) = 1 \quad (79)$$

with general solution

$$T = -\frac{x^2}{2} + Ax + B. \quad (80)$$

However, the boundary conditions now are

$$\frac{dT}{dx}\Big|_{x=0} = 0, \quad T(1) = 1. \quad (81)$$

This is because there is no heat source/sink at $x = 0$ meaning that the heat flux $-dT/dx$ must be zero there. These conditions imply

$$\frac{dT}{dx}\Big|_{x=0} = A = 0 \quad (82)$$

and

$$T(1) = -\frac{1}{2} + B = 1 \implies B = \frac{3}{2} \quad (83)$$

giving

$$T = -\frac{x^2}{2} + \frac{3}{2}. \quad (84)$$

- (e) The heat flux *leaving* the rod at $x = 1$ for the solution in part (e) is

$$-\frac{dT}{dx}\Big|_{x=1} = 1. \quad (85)$$

[Note: now all the electrical energy is leaving the rod (as heat) at this end].

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Review of mark distribution:

Total A marks: 32 of 32 marks

Total B marks: 20 of 20 marks

Total C marks: 12 of 12 marks

Total D marks: 16 of 16 marks

Total Mastery marks: 0 of 20 marks

Total marks: 80 of 80 marks

- The set of linearly independent vectors

$$\mathbf{x}_k = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \vdots \\ \omega^{(n-1)k} \end{pmatrix}, \quad \omega = e^{2\pi i/n}, \quad k = 0, 1, \dots, n-1,$$

are complex-valued eigenvectors of an n -by- n circulant matrix.

- The matrix

$$\mathbf{K}_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}, \quad n \geq 2$$

has orthonormal eigenvectors $\{\Phi_m\}$ and corresponding eigenvalues $\{\lambda_m\}$ given by

$$\Phi_m = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{m\pi}{n+1}\right) \\ \sin\left(\frac{2m\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{nm\pi}{n+1}\right) \end{pmatrix}, \quad \lambda_m = 2 - 2\cos\left(\frac{\pi m}{n+1}\right), \quad m = 1, \dots, n$$

- The current density $J(x)$ in a 1D conductor with conductivity distribution $C(x)$ along the x -axis is related to the voltage distribution $\Phi(x)$ and the external current source density distribution $F(x)$ by

$$J(x) = -C(x) \frac{d\Phi(x)}{dx}, \quad \frac{dJ}{dx} = F(x).$$

- The current density vector $\mathbf{j} = (J^{(x)}(x, y), J^{(y)}(x, y))^T$ in a 2D conductor with uniform conductivity $\hat{c}$ located in an (x, y) -plane is related to the voltage distribution $\Phi(x, y)$ and the external current source density distribution $F(x, y)$ in the conductor by

$$\mathbf{j}(x, y) = \begin{pmatrix} J^{(x)}(x, y) \\ J^{(y)}(x, y) \end{pmatrix} = -\hat{c} \nabla \Phi(x, y), \quad \nabla \cdot \mathbf{j} = F(x, y).$$

The voltage potential $\Phi(x, y)$ in a two-dimensional conductor of uniform conductivity $\hat{c}$ can be written in terms of an analytic function $h(z)$,

$$\Phi(x, y) = \operatorname{Re}[h(z)], \quad z = x + iy,$$

where $\operatorname{Re}[\cdot]$ denotes the real part of the quantity in square brackets. Also,

$$J^{(x)}(x, y) - iJ^{(y)}(x, y) = -\hat{c} \frac{dh}{dz}.$$

If, near some point z_0 inside the conductor, $h(z)$ has a local form

$$h(z) = -\frac{m}{2\pi} \log(z - z_0) + \hat{h}(z),$$

where $\hat{h}(z)$ is non-singular at z_0 , then we say there is a point source of current of strength m at z_0 .